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Introduction to Green-Functions in Many-Body-Physics in (or close to) Thermodynamic Equilibrium

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Basic Setting:

$$\mathscr{F}_\zeta = \mathrm{span}_{\mathbb{C}} \left\{ |n_1, n_2, \ldots, n_{\mathrm{d}}\rangle \,\middle|\, n_1, n_2, \ldots, n_{\mathrm{d}} \in \mathbb{N}_\zeta \right\} \qquad \mathbb{N}_\zeta = \begin{cases} \{0,1\} & \text{for } \zeta = -1 \\ \{0,1,2,\ldots,\infty\} & \text{for } \zeta = +1 \end{cases}$$

$$H = \sum_{i,j} H_{ij} c_i^\dagger c_j \quad + \quad \sum_{ijkl} V_{ijkl}\, c_i^\dagger c_j^\dagger c_k c_l \qquad \left[c_i, c_j\right]_\zeta = \left[c_i^\dagger, c_j^\dagger\right]_\zeta = 0, \qquad \left[c_i, c_j^\dagger\right]_\zeta = \delta_{i,j}$$

grand-canonical-ensemble with natural variables (T, V, μ).

Fact 4.3 In the grand-canonical-ensemble, expectation-values are evaluated w.r.t. the state

$$\rho = \frac{\mathrm{e}^{-\beta(H-\mu N)}}{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}} \equiv \frac{1}{Z} \mathrm{e}^{-\beta(H-\mu N)},$$

Exception: SSB!
E.g. BCS-Theory

which solves the stationary von Neumann equation.

Remark 4.1.1 Usually one switches the notation to a more generic case, and just writes generalized (system-specific) indices $\left\{c_1, c_1^\dagger, c_2, c_2^\dagger, \ldots, c_{\mathrm{d}}, c_{\mathrm{d}}^\dagger\right\} \equiv \left\{c_\alpha, c_\alpha^\dagger\right\}_{\alpha \in \mathscr{A}}$.

Example: Electrons in a Crystal

The Hamiltonian of a free electron-gas in a crystal with quartic interactions reads

$$H \equiv \sum_{\substack{\mathbf{k}\in Q \\ \sigma\in\{\downarrow,\uparrow\} \\ n\in\{1,\ldots,\infty\}}} \epsilon_{n\sigma\mathbf{k}}\, c^{\dagger}_{n\sigma\mathbf{k}} c_{n\sigma\mathbf{k}} \quad + \sum_{\substack{n_1 n_2 n_3 n_4 \\ \sigma_1\sigma_2\sigma_3\sigma_4 \\ \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}} V_{\substack{n_1 n_2 n_3 n_4 \\ \sigma_1\sigma_2\sigma_3\sigma_4 \\ \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}}\, c^{\dagger}_{n_1\sigma_1\mathbf{k}_1} c^{\dagger}_{n_2\sigma_2\mathbf{k}_2} c_{n_3\sigma_3\mathbf{k}_3} c_{n_4\sigma_4\mathbf{k}_4}$$

$$\equiv \sum_{\alpha\in\mathscr{A}} \epsilon_{\alpha} c^{\dagger}_{\alpha} c_{\alpha} \quad + \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} V_{\alpha_1\alpha_2\alpha_3\alpha_4} c^{\dagger}_{\alpha_1} c^{\dagger}_{\alpha_2} c_{\alpha_3} c_{\alpha_4}$$

grand-canonical-ensemble with natural variables (T, V, μ). $\rho = \dfrac{\mathrm{e}^{-\beta(H-\mu N)}}{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}} \equiv \dfrac{1}{Z}\mathrm{e}^{-\beta(H-\mu N)}$,

- **Temperature**: Controlled by the experiment
- **Volume**: Dictated by the size of the crystal / lattice constant (corresponds to the distance between the wave-vectors), can also be controlled by applying pressure to the system.
- **Chemical Potential**: Either dictated by the number of electrons, or by other means (e.g. an external electric field can be used to control the chemical potential)

Quote-Collection on the Grand Canonical Ensemble (Mahan, Negele & Orland, many more in my lists…):

Its thermal average is obtained simply by taking the thermal average of n_λ:

$$\langle N \rangle = \sum_\lambda \langle n_\lambda \rangle = \sum_\lambda N_\lambda \qquad (1.121)$$

This equation serves as a definition of the chemical potential and determines its variations with temperature and particle number N.

N is proportional to $e^{-\frac{(E-\mu N)}{kT}}$. In addition to the average energy of the system, the average particle number is now fixed and is controlled by the chemical potential μ.

At finite temperature, the energy is given by $E = \Omega + \mu N + TS$, where the chemical potential μ must be adjusted such that $N = -\frac{\partial \Omega}{\partial \mu}$ yields the desired particle number.

The finite temperature expansion contains $(\mu - \mu_0)N$ where the chemical potential for the non-interacting system, defined such that $\frac{\partial \Omega_0}{\partial \mu}\,|_{\mu_0} = N$, need not agree with the chemical potential of the interacting system defined by $\frac{\partial \Omega}{\partial \mu}\,|_{\mu} = N$.

Definition 6.1 For a multi-index $A \equiv (\alpha, t) \in \mathscr{A} \times \mathbb{R} \equiv I_t$ we define the *real-time-one-body-Green-function* in its various shapes

$$
\begin{aligned}
G^{++}(A, A') \equiv G^{\mathbb{T}}(A, A') \equiv G(A, A') &\equiv G_{\alpha,\alpha'}(t, t') \\
&\equiv -\frac{\mathrm{i}}{\hbar}\langle \mathrm{T}_t\, a_\alpha(t)\, a^\dagger_{\alpha'}(t')\rangle \\
&\equiv -\frac{\mathrm{i}}{\hbar}\frac{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}\, \mathrm{T}_t\, a_\alpha(t)\, a^\dagger_{\alpha'}(t')}{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}} \\
&\equiv -\frac{\mathrm{i}}{\hbar}\frac{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}\left[\theta(t-t')\, a_\alpha(t)\, a^\dagger_{\alpha'}(t') + \zeta\, \theta(t'-t)\, a^\dagger_{\alpha'}(t')\, a_\alpha(t)\right]}{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}}
\end{aligned}
$$

with $\quad a_\bullet(t) = \mathrm{e}^{\mathrm{i}t(H-\mu N)/\hbar} a_\bullet \mathrm{e}^{-\mathrm{i}t(H-\mu N)/\hbar}$

$$
\begin{aligned}
G^{++}(A, A') \equiv G^{\mathbb{T}}(A, A') \equiv G(A, A') &\equiv G_{\alpha,\alpha'}(t, t') \\
&\equiv -\frac{\mathrm{i}}{\hbar}\langle \mathrm{T}_t\, a_\alpha(t)\, a^\dagger_{\alpha'}(t')\rangle \\
&\equiv -\frac{\mathrm{i}}{\hbar}\frac{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}\, \mathrm{T}_t\, a_\alpha(t)\, a^\dagger_{\alpha'}(t')}{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}} \\
&\equiv -\frac{\mathrm{i}}{\hbar}\frac{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}\left[\theta(t-t')\, a_\alpha(t)\, a^\dagger_{\alpha'}(t') + \zeta\, \theta(t'-t)\, a^\dagger_{\alpha'}(t')\, a_\alpha(t)\right]}{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}}
\end{aligned}
$$

Interpretation:

The one-particle Green's function describes a Gedankenexperiment which can never happen in practice. One creates a particle at time t' in the state α' and then (if $t > t'$) annihilates another particle at time t in the state α.

Fact: Many Observable Quantities that can be expressed through Green-functions!!!

- (Local) Magnetization → “lesser GF”
- (Local) Density of States → “retarded GF”
- (Local) Density of Occupied States → “lesser GF”
- Specific Heat → “lesser GF”
- Pressure → “imaginary time / Matsubara GF”
- (Entropy) → “imaginary time / Matsubara GF”
- Polarization → “lesser GF”
- “Any” (e.g. (Tunneling) Electric Currents with Kubo-Formula) → “retarded GF”

More One-Body-Green-Functions:

- $G(A, A') \equiv G^{++}(A, A') \equiv G^{\mathbb{T}}(A, A') \equiv -\frac{\mathrm{i}}{\hbar}\langle \mathrm{T}_t\, a_\alpha(t) a^\dagger_{\alpha'}(t')\rangle$
- $G^{+-}(A, A') \equiv G^{<}(A, A') \equiv -\frac{\mathrm{i}}{\hbar}\zeta\langle a^\dagger_{\alpha'}(t') a_\alpha(t)\rangle$
- $G^{-+}(A, A') \equiv G^{>}(A, A') \equiv -\frac{\mathrm{i}}{\hbar}\langle a_\alpha(t) a^\dagger_{\alpha'}(t')\rangle$
- $G^{--}(A, A') \equiv G^{\overline{\mathbb{T}}}(A, A') \equiv -\frac{\mathrm{i}}{\hbar}\langle \overline{\mathrm{T}}_t a_\alpha(t) a^\dagger_{\alpha'}(t')\rangle$
- $G^{\mathrm{ret}}(A, A') = -\frac{\mathrm{i}}{\hbar}\theta(t - t')\langle \left[a_\alpha(t),\, a^\dagger_{\alpha'}(t')\right]_\zeta\rangle$
- $G^{\mathrm{adv}}(A, A') = +\frac{\mathrm{i}}{\hbar}\theta(t' - t)\langle \left[a_\alpha(t),\, a^\dagger_{\alpha'}(t')\right]_\zeta\rangle$

Remark 6.1.4 With the cyclicity of the trace, and the Baker-Campbell-Hausdorff-formula, one can conclude, that all the Green-functions defined above only depend on the time-difference.

Example: A Diagonalized, Non-Interacting Electron-Gas

$$H = \sum_{\alpha} \epsilon_{\alpha} c_{\alpha}^{\dagger} c_{\alpha}$$

Theorem 6.3 Introducing the variables $\xi_{\alpha} \equiv \epsilon_{\alpha} - \mu$ it follows

$$G^{\mathbb{T}}_{\alpha,\alpha'}(t) = +\frac{\mathrm{i}}{\hbar}\,\delta_{\alpha,\alpha'}\,\mathrm{e}^{-\mathrm{i}\xi_{\alpha}t/\hbar}\left(\mathrm{F}_{\beta}(\xi_{\alpha}) - \theta(t)\right) \qquad G^{\mathbb{T}}_{\alpha,\alpha'}(\omega \pm \mathrm{i}\delta) = \text{TBD}$$

$$G^{<}_{\alpha,\alpha'}(t) = +\frac{\mathrm{i}}{\hbar}\,\delta_{\alpha,\alpha'}\,\mathrm{e}^{-\mathrm{i}\xi_{\alpha}t/\hbar}\,\mathrm{F}_{\beta}(\xi_{\alpha}) \qquad G^{<}_{\alpha,\alpha'}(\omega) = +\frac{\mathrm{i}}{\hbar}\,\delta_{\alpha,\alpha'}\,2\pi\,\delta(\omega - \xi_{\alpha}/\hbar)\,\mathrm{F}_{\beta}(\xi_{\alpha})$$

$$G^{>}_{\alpha,\alpha'}(t) = -\frac{\mathrm{i}}{\hbar}\,\delta_{\alpha,\alpha'}\,\mathrm{e}^{-\mathrm{i}\xi_{\alpha}t/\hbar}\left(1 - \mathrm{F}_{\beta}(\xi_{\alpha})\right) \qquad G^{>}_{\alpha,\alpha'}(\omega) = -\frac{\mathrm{i}}{\hbar}\,\delta_{\alpha,\alpha'}\,2\pi\,\delta(\omega - \xi_{\alpha}/\hbar)\left(1 - \mathrm{F}_{\beta}(\xi_{\alpha})\right)$$

$$G^{\overline{\mathbb{T}}}_{\alpha,\alpha'}(t) = \text{TBD} \qquad G^{\overline{\mathbb{T}}}_{\alpha,\alpha'}(\omega \pm \mathrm{i}\delta) = \text{TBD}$$

$$G^{\mathrm{ret}}_{\alpha,\alpha'}(t) = -\frac{\mathrm{i}}{\hbar}\delta_{\alpha,\alpha'}\,\mathrm{e}^{-\mathrm{i}\xi_{\alpha}t/\hbar}\theta(t) \qquad G^{\mathrm{ret}}_{\alpha,\alpha'}(\omega + \mathrm{i}\delta) = \frac{1}{\hbar}\frac{\delta_{\alpha,\alpha'}}{\omega + \mathrm{i}\delta - \xi_{\alpha}/\hbar}$$

$$G^{\mathrm{adv}}_{\alpha,\alpha'}(t) = +\frac{\mathrm{i}}{\hbar}\delta_{\alpha,\alpha'}\,\mathrm{e}^{-\mathrm{i}\xi_{\alpha}t/\hbar}\theta(-t) \qquad G^{\mathrm{a}}_{\alpha,\alpha'}(\omega - \mathrm{i}\delta) = \frac{1}{\hbar}\frac{\delta_{\alpha,\alpha'}}{\omega - \mathrm{i}\delta - \xi_{\alpha}/\hbar}$$

Fact:

For Interacting Systems (which have more than a few energy-levels) it is impossible to analytically obtain the various Green-functions

Cure (for “weak” interactions):

Introducing…

1. another (!) Green-Function
2. some math.

Definition 7.1 For a multi-index $A \equiv (\alpha, \tau) \in \mathscr{A} \times \mathbb{R}$ we define the *imaginary-time one-body Green-function* in its various shapes

$$\begin{aligned}\mathscr{G}(A, A') \equiv \mathscr{G}_{\alpha,\alpha'}(\tau, \tau') &\equiv -\frac{1}{\hbar}\langle \mathrm{T}_\tau\, a_\alpha(\tau)\, a^\dagger_{\alpha'}(\tau')\rangle \\ &\equiv -\frac{1}{\hbar}\frac{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}\, \mathrm{T}_\tau\, a_\alpha(\tau)\, a^\dagger_{\alpha'}(\tau')}{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}} \\ &\equiv -\frac{1}{\hbar}\frac{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}\Big[\theta(\tau-\tau')\, a_\alpha(\tau)\, a^\dagger_{\alpha'}(\tau') + \zeta\, \theta(\tau'-\tau)\, a^\dagger_{\alpha'}(\tau')\, a_\alpha(\tau)\Big]}{\mathrm{Tr}_{\mathscr{F}_\zeta}\, \mathrm{e}^{-\beta(H-\mu N)}}\end{aligned}$$

with $\quad a_\bullet(\tau) = \mathrm{e}^{\tau(H-\mu N)/\hbar} a_\bullet \mathrm{e}^{-\tau(H-\mu N)/\hbar}, \qquad a^\dagger_\bullet(\tau) = \mathrm{e}^{\tau(H-\mu N)/\hbar} a^\dagger_\bullet \mathrm{e}^{-\tau(H-\mu N)/\hbar}.$

Theorem 7.2 It follows

$$\mathscr{G}_{\alpha,\alpha'}(\tau, \tau') \equiv \mathscr{G}_{\alpha,\alpha'}(\tau - \tau', 0) \equiv \mathscr{G}_{\alpha,\alpha'}(\tau - \tau') \equiv \zeta \mathscr{G}_{\alpha,\alpha'}(\tau - \tau' + \hbar\beta),$$

which implies that the function

$$(-\hbar\beta, \hbar\beta) \ni \tau \mapsto \mathscr{G}_{\alpha,\alpha'}(\tau) \in \mathbb{C}$$

is periodic.

Lemma 2.3 By a Fourier-expansion one finds the expansion for the periodic standard-continuation as

$$\mathscr{G}_\zeta(\tau) = \frac{1}{\hbar\beta} \sum_{\omega_n \in \Omega_\zeta} \mathscr{G}_\zeta(\mathrm{i}\omega_n)\, \mathrm{e}^{-\mathrm{i}\omega_n \tau}$$

with coefficients

$$\mathscr{G}_\zeta(\mathrm{i}\omega_n) = \int_0^{\hbar\beta} \mathrm{d}\tau\, \mathscr{G}_\zeta(\tau)\, \mathrm{e}^{\mathrm{i}\omega_n \tau}.$$

with Matsubara-frequencies that depend on the statistics

$$\Omega_\zeta = \frac{2\mathbb{Z} + 1 - \theta(\zeta)}{\hbar\beta}\pi.$$

$$\mathbb{Z} \ni n \mapsto \omega_n = \frac{2n + 1 - \theta(\zeta)}{\hbar\beta}\pi = \begin{cases} 2n\pi/\hbar\beta & \text{for bosons} \\ (2n+1)\pi/\hbar\beta & \text{for fermions} \end{cases}$$

Example: A Diagonal Gas $H = \sum_{\alpha} \epsilon_\alpha c_\alpha^\dagger c_\alpha$ introducing the variable $\xi_\alpha \equiv \epsilon_\alpha - \mu$,



$$\mathcal{G}_{\alpha\alpha'}(\tau) = -\frac{1}{\hbar} e^{-\tau\xi_\alpha/\hbar} \delta_{\alpha,\alpha'} \left[\theta(\tau)\left(1 + \zeta\, n_\beta^\zeta(\xi_\alpha)\right) + \theta(-\tau)\zeta\, n_\beta^\zeta(\xi_\alpha)\right].$$



$$\mathcal{G}_{\alpha,\alpha'}(i\omega_n) = \frac{1}{\hbar} \frac{\delta_{\alpha,\alpha'}}{i\omega_n - \xi_\alpha/\hbar}$$

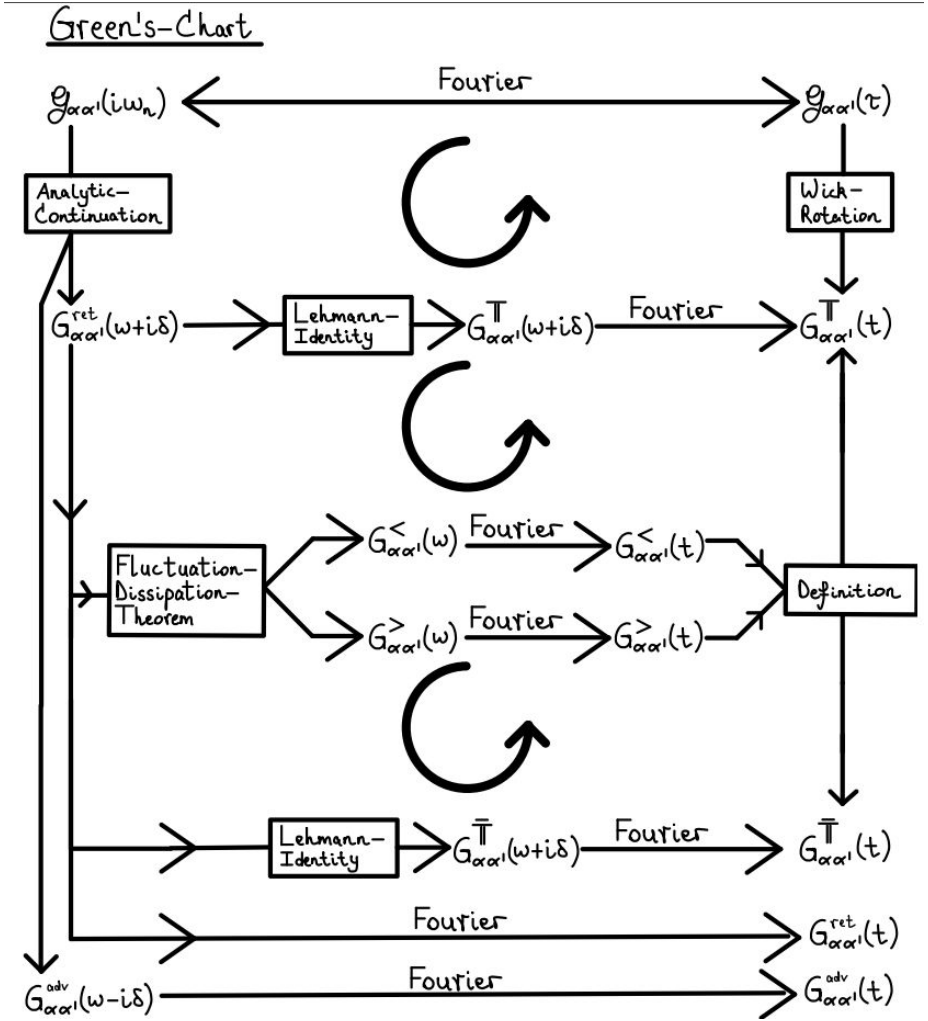
Example: A Non-Interacting Statistics-Neutral-Gas

$$H = \sum_{\alpha,\beta} H_{\alpha,\beta} c_\alpha^\dagger c_\beta \equiv \mathbf{c}^\dagger \cdot \mathbf{H} \cdot \mathbf{c}$$

$$\mathscr{G}_{0\ \alpha,\alpha'}(\mathrm{i}\omega_n) = \frac{1}{\hbar}\left(\left[\mathrm{i}\omega_n - \frac{H-\mu}{\hbar}\right]^{\mathrm{inv}}\right)_{\alpha,\alpha'} \quad \Longleftrightarrow: \quad \mathscr{G}_0(\mathrm{i}\omega_n) \equiv \frac{1}{\hbar}\frac{1}{\mathrm{i}\omega_n - \frac{H-\mu}{\hbar}}$$

$$\mathscr{G}_{0\ \alpha,\alpha'}(\tau) = \frac{1}{\hbar\beta}\sum_{\omega_n \in \Omega_\zeta} \mathscr{G}_{0\ \alpha,\alpha'}(\mathrm{i}\omega_n)\, \mathrm{e}^{-\mathrm{i}\omega_n \tau}$$





holds for

- **non-interacting systems**

AND for

- **interacting systems!**

With the so-called “Lehmann-Representation”:

analytic continuation

$$\begin{aligned} \mathrm{i}\omega_n \mapsto \omega + \mathrm{i}\delta \quad &\Rightarrow \quad \mathscr{G}_{\alpha,\alpha'}(\mathrm{i}\omega_n) \mapsto G^{\mathrm{ret}}_{\alpha,\alpha'}(\omega + \mathrm{i}\delta) \\ \mathrm{i}\omega_n \mapsto \omega - \mathrm{i}\delta \quad &\Rightarrow \quad \mathscr{G}_{\alpha,\alpha'}(\mathrm{i}\omega_n) \mapsto G^{\mathrm{adv}}_{\alpha,\alpha'}(\omega - \mathrm{i}\delta) \end{aligned}$$

The *fluctuation-dissipation-theorem*[1] states the two identities

$$\begin{aligned} G^{<}_{\alpha\alpha'}(\omega) &= 2\mathrm{i}\,\zeta\,\mathrm{n}^{\zeta}_{\hbar\beta}(\omega)\,\mathrm{Im}\lim_{\eta\to 0} G^{\mathrm{ret}}_{\alpha\alpha'}(\omega + \mathrm{i}\eta), \\ G^{>}_{\alpha\alpha'}(\omega) &= 2\mathrm{i}\left(1 + \zeta\,\mathrm{n}^{\zeta}_{\hbar\beta}(\omega)\right)\mathrm{Im}\lim_{\eta\to 0} G^{\mathrm{ret}}_{\alpha\alpha'}(\omega + \mathrm{i}\eta). \end{aligned}$$

$$\mathrm{F}_{\hbar\beta}(\omega) = \frac{1}{\mathrm{e}^{\hbar\beta\omega} + 1} \quad / \quad \mathrm{B}_{\hbar\beta}(\omega) = \frac{1}{\mathrm{e}^{\hbar\beta\omega} - 1} \quad / \quad \mathrm{n}^{\zeta}_{\hbar\beta}(\omega) = \frac{1}{\mathrm{e}^{\hbar\beta\omega} - \zeta}$$

$$\zeta \in \{-1, +1\}$$

Systematic Calculation of Imaginary-Time One-Body Green-Functions for Statistics-Neutral Interacting Systems

$$\sum_{i,j} H_{ij} c_i^\dagger c_j \quad + \quad \sum_{ijkl} V_{ijkl}\, c_i^\dagger c_j^\dagger c_k c_l \qquad\qquad \mathcal{G}_0(\mathrm{i}\omega_n) \equiv \frac{1}{\hbar} \frac{1}{\mathrm{i}\omega_n - \frac{H-\mu}{\hbar}}$$

Notation 9.9 For reasons that will become clear in a later chapter[3] we assume[4] the existence of a function[5]

$$\begin{aligned} \Sigma:\ \mathrm{i}\Omega_\xi &\longrightarrow \mathrm{Mat}_{\mathrm{d}\times\mathrm{d}}(\mathbb{C}) \\ \mathrm{i}\omega_n &\longmapsto \Sigma(\mathrm{i}\omega_n) \end{aligned}$$

s.t. the *Dyson-equation* the (connected or unconnected) one-body Green-function is fulfilled

$$\mathcal{G}(\mathrm{i}\omega_n) = \mathcal{G}_0(\mathrm{i}\omega_n) + \left(\mathcal{G}_0 \cdot \Sigma \cdot \mathcal{G}\right)(\mathrm{i}\omega_n),$$

where the dot inside the parenthesis corresponds to matrix-multiplication.

Self-Energy…?

$$\sum_{i,j} H_{ij} c_i^\dagger c_j \quad + \quad \sum_{ijkl} V_{ijkl}\, c_i^\dagger c_j^\dagger c_k c_l \qquad\qquad \mathcal{G}_0(\mathrm{i}\omega_n) \equiv \frac{1}{\hbar} \frac{1}{\mathrm{i}\omega_n - \frac{H-\mu}{\hbar}}$$

Dyson-equation $\mathcal{G}(\mathrm{i}\omega_n) = \mathcal{G}_0(\mathrm{i}\omega_n) + \left(\mathcal{G}_0 \cdot \Sigma \cdot \mathcal{G}\right)(\mathrm{i}\omega_n).$

Remark 9.9.3 The justification for the name „self-energy“ can be seen in the following way: A formal inversion of the equation above yields

$$\mathcal{G}(\mathrm{i}\omega_n) = \left(1 - \mathcal{G}_0 \cdot \Sigma\right)^{-1}(\mathrm{i}\omega_n) \cdot \mathcal{G}_0(\mathrm{i}\omega_n) = \frac{1}{\left(\mathcal{G}_0^{-1} - \Sigma\right)(\mathrm{i}\omega_n)} = \frac{1}{\hbar} \frac{1}{\mathrm{i}\omega_n - \left(H + \Sigma(\mathrm{i}\omega_n) - \mu\right)/\hbar} \tag{9.3}$$

The self-energy is therefore a term, that is formally being added on top of the Hamiltonian-matrix.

Methods to compute the self-energy:

- Equation of motion technique
- S-matrix-expansion (with or without functional integral formalism)
- combination of the first two
- Functional Renormalization
- Self-consistent numerical techniques

Not discussed, but also relevant in this context:

There are more (Families of) Green-Functions …

- n-body-Green-functions
- Connected Green-functions
- Generalized Green-Functions
- Phononic and Photonic Displacement Green-Functions
- Zero-Temperature-Limit / Spontaneous Symmetry Breaking / Non-Equilibrium /
- Basis-Change (...)

… with similar equations and identities.

There are more Concepts …

- Feynman-Diagramology (?)
- Functional Field Integral Formalism
- Symmetry-Principles (?)

Summary:

- **Basic idea:** Describe systems (and observables) with matrices (e.g.: one-body-Matsubara-GF) or matrix-valued functions (e.g.: all other one-body-GF)

- There are different **Families** of Green-functions. Within a family, all GFs are connected to each other. E.g.: **Knowledge of one** (one-body) Green-function **implies the knowledge of every other** (one-body) Green-function

- Green-Functions are **one of the most important tools** to study **interacting** systems at finite temperatures.