Functional Renormalization for a Fermionic System with a Topological Phase Transition away from Thermodynamic Equilibrium

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1 Introduction

In the 1950s and 1960s, it was experimentally recognized that completely different systems can behave very similar in the vicinity of phase transitions. This similarity is quantified by the comparison of system-specific numbers, so-called "critical exponents". It is said that two systems are in the same "universality class" if they have (approximately) the same critical exponents.

Since this discovery, people have been investigating this phenomenon and trying to answer the questions:

1. How many universality classes are there?
2. How can we theoretically calculate the critical exponents?
3. Why are the critical exponents of different systems sometimes so similar?

The aim of this master thesis is to determine the critical exponents of a concrete system, which has a topological phase transition.

Thus, we strive to take a small step towards answering the first question by resorting to concepts (the functional integral formalism and the functional renormalization group (FRG)), which have been found in the course of time to answer the second question. The interesting thing about this work is that we address a concept like "universality" for fermionic systems *away from thermodynamic equilibrium* (contrary to the already known universality classes).

This thesis is designed in such a way that students of physics, having completed their undergraduate studies and having had an introduction to the field theory of many-particle systems (roughly within the scope of the book of A. Altland [1]) will be able to read and understand this work without further literature.

Chapter 2, "Fundamentals", briefly reviews the topics that are assumed to be known.

Chapter 3, "Physical Concepts and Mathematical Methods", introduces the tools that go beyond the basic lectures which are needed for our analysis of the concrete system in the subsequent chapters 4, 5 and 6. In particular, concepts from the templates [5], [6], [7], [8] are generalized[1] to arbitrary fermionic systems and processed, so that one can easily reference and use the results (in later chapters, but also in future work). Furthermore the equivalence to Wilson's renormalization scheme is explained.

In chapter 4 a concrete system from [15] is introduced, which has a formal correspondence to a topological insulator. Just like the underlying topological insulator, as it will turn out in the following chapter, this system also has a phase transition, which we want to investigate.

In chapter 5, for reasons that are difficult to motivate in advance, we will first use seemingly different but mutually equivalent approximation-schemes from [16] and [17] to determine the self-energy of our system in first order perturbation theory. Furthermore, the validity of these approximations is investigated a posteriori by contrasting control results. Correlators of all kinds are computed, the divergent corrections to the "One-Loop-Approximation" are determined (these are the only corrections that we are really interested in, which will be explained in chapter

[1] In the templates there is, e.g., always some sort of momentum dependence assumed.

3) and pictographically presented (with Feynman-diagrams) and finally the divergent corrections are subjected to another “sanity-check” to make sure that we did not miscalculate.

In chapter 6 we will use the remaining tools from chapter 3 for the functional renormalization of the concrete system to write down and solve flow equations.

Finally, in chapter 7, the key messages of the thesis are summarized.

To make the structure of the thesis more compact and to facilitate readability, computationally intensive proofs have been moved to the appendix.

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2 Fundamentals

This chapter briefly lists basic terms and concepts from [1] that are assumed to be known and therefore are not further substantiated or proven.

2.1 Second Quantization and Graßmann-Numbers

Notation 2.1 Let $\mathcal{H} = \text{span}_{\mathbb{C}}\{|1\rangle, |2\rangle, \ldots\}$ be a single-particle Hilbert-space with orthonormal basis. The orthonormal, antisymmetrized occupation number states of the corresponding fermionic Fock-space $\mathscr{H} \equiv \mathscr{F}(\mathcal{H})$ are then given by

$$\prod_{j=1}^{N} a_{i_j}^{\dagger} \mid \text{vac.} \rangle = \frac{1}{\sqrt{N!}} \sum_{\pi \in \mathrm{S}_N} \text{sgn}(\pi) \bigotimes_{j=1}^{N} |i_{\pi(j)}\rangle,$$

with $\mid \text{vac.} \rangle \equiv 1 \in \mathbb{C}$, where the creation- and annihilation-operators $\{a_1, a_1^{\dagger}, a_2, a_2^{\dagger}, \ldots\}$ obey the CAR ("[C]anonical [A]nticommutation [R]elations") .

$$\{a_i, a_j\} = \{a_i^{\dagger}, a_j^{\dagger}\} = 0, \qquad \{a_i, a_j^{\dagger}\} = \delta_{i,j}.$$

Notation 2.2 In one-to-one-correspondence to the orthonormal basis $\{|1\rangle, |2\rangle, \ldots\}$ one introduces anticommuting Graßmann-numbers $\{\psi_1, \overline{\psi}_1, \psi_2, \overline{\psi}_2, \ldots\}$ and coherent states

$$\begin{aligned} \mid \psi \rangle &= \exp\left(- \sum_i \psi_i a_i^{\dagger} \right) \mid \text{vac.} \rangle, \\ \langle \psi \mid &= \langle \text{vac.} \mid \exp\left(- \sum_i a_i \overline{\psi}_i \right). \end{aligned}$$

These form an overcomplete set of states according to which one can expand elements of the Fock-space, with the help of the Berezin-integral[1]:

$$\mathbb{1} \equiv \mathbb{1}_{\mathscr{H}} = \int \mathrm{d}\psi \, \mathrm{e}^{-|\psi|^2} \mid \psi \rangle\langle \psi \mid, \tag{2.1}$$

[1] It is also said that a "resolution of the identity" is obtained.

where

$$\mathrm{d}\psi \equiv \mathrm{d}(\psi, \overline{\psi}) = \prod_i \mathrm{d}\overline{\psi}_i \mathrm{d}\psi_i \quad \text{und} \quad |\psi|^2 \equiv \overline{\psi} \cdot \psi \equiv \sum_i \overline{\psi}_i \psi_i. \tag{2.2}$$

Remark 2.2.1 It is helpful to note that Graßmann-numbers always implicitly correspond to orthonormal basis vectors with whose creation- and annihilation-operators they anticommute. In the later application, starting from a one-dimensional lattice with Wannier-states, we will make a unitary change of basis, corresponding to a change of the fermionic creation- and annihilation-operators $\{a_1, a_1^\dagger, a_2, a_2^\dagger, \ldots\}$ to new fermionic creation- and annihilation-operators $\{d_1, d_1^\dagger, d_2, d_2^\dagger, \ldots\}$. These operators then also yield new Graßmann-numbers $\{\xi_1, \overline{\xi}_1, \xi_2, \overline{\xi}_2, \ldots\}$.

Notation 2.3 Let L be a normal ordered operator

$$L = \sum_{\substack{i_1 < \ldots < i_n \\ j_1 < j_2 < \ldots < j_{m-1} < j_m}} L^{i_1,\ldots,i_n}_{j_1,\ldots,j_m} a^\dagger_{i_1} \cdots a^\dagger_{i_n} a_{j_1} \cdots a_{j_m},$$

with $L^{i_1,\ldots,i_n}_{j_1,\ldots,j_m} \in \mathbb{C}$ (omitting units for notational simplicity). Then we write for the overlap

$$\langle\, \psi \mid L \mid \psi' \,\rangle = L(\overline{\psi}, \psi') \langle\, \psi \mid \psi' \,\rangle,$$

with

$$L(\overline{\psi}, \psi') = \sum_{\substack{i_1 < \ldots < i_n \\ j_1 < j_2 < \ldots < j_{m-1} < j_m}} L^{i_1,\ldots,i_n}_{j_1,\ldots,j_m} \overline{\psi}_{i_1} \cdots \overline{\psi}_{i_n} \psi'_{j_1} \ldots \psi'_{j_m}.$$

Remark 2.3.1 For operators that are not normally ordered, we adopt this notation and implicitly require that a normal ordering is performed, where the additional terms arising in the normal ordering are included in the symbol $L(\overline{\psi}, \psi')$.

3 Physical Concepts and Mathematical Methods

While the previous chapter merely repeated terms and contents that were presupposed to be known, this chapter introduces physical concepts and mathematical methods, which mostly go beyond the basic lectures and will be needed in the subsequent analysis of a concrete system.

3.1 Fermionic Open Quantum System

3.1.1 Lindblad-Equation

Microscopic Derivation

One considers a system consisting of a subsystem which one wants to investigate and a bath which interacts with the subsystem. It is well known from introductory lectures about physics, that the subsystem is described by a

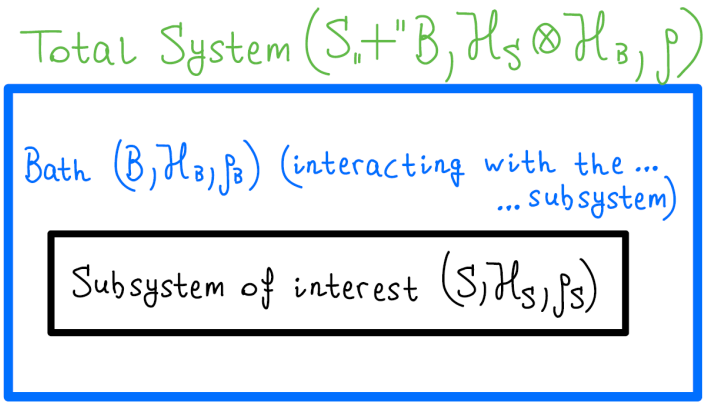


Figure 3.1: The picture visualizes the situation.

density ρ_S, which is obtained by "tracing out the bath" which we symbolically denote by

$$\rho_S = \mathrm{Tr}_{\mathrm{Bath}}\, \rho.$$

After several assumptions, approximations and calculations[1] one arrives at the Lindblad-equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_S \approx -\frac{\mathrm{i}}{\hbar}[H, \rho_S] + \sum_\alpha \kappa_\alpha \left(2L_\alpha \rho_S L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho_S\}\right).$$

[1] See, for example, [2] or [3] for a detailed derivation.

Within this thesis we will only consider systems whose time evolution is dominated by the dissipative part. Furthermore, for notational simplicity, we drop the index S from now on and include the rate κ_α into the definition of the quantum jump operators L_α, which will not generate confusion at any point in the following. This leads to a simplification of the Lindblad-equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho \approx \sum_\alpha 2L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\} \equiv \mathscr{L}[\rho].$$

Remark Such a microscopic derivation is presented in literature as [2] and [3]. The quantum jump operators L_α are then defined by the interaction between subsystem and environment. In the last century, however, symmetry principles in particular have proven to be very good guides for derivations of equations[2]. It is exactly this guide that we will follow in this thesis when we introduce our particular system: We will present a starting situation and express a rough idea of what the interactions between the subsystem and the bath should be, will require particle number conservation and dictate the shape of a steady state, and from this we will *define* quantum jump operators.

3.1.2 From the Lindblad-Equation to Quantum Field Theory

> *"Quantum field theory (QFT) provides an extremely powerful set of computational methods that have yet to find any fundamental limitations. It has led to the most fantastic agreement between theoretical predictions and experimental data in the history of science. It provides deep and profound insights into the nature of our universe, and into the nature of other possible self-consistent universes. On the other hand, the subject is a mess."*
>
> M. D. Schwartz in [4], p. xv

The remainder of this chapter is mainly devoted to the introduction and generalization of methods and tools (that mostly go beyond the basic lectures) from the various literatures and templates [5] - [12], unifying different concepts and prefactors-conventions into a common notation.

Translation of the Partition Function into a Functional Integral

Theorem 3.1 A Lindblad-equation[3]

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = \sum_\alpha 2L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\} \equiv \mathscr{L}[\rho] \tag{3.1}$$

has the formal solution

$$\rho(t) = \exp\Big[(t - t_\mathrm{i})\mathscr{L}[\bullet]\Big]\,\rho(t_\mathrm{i}). \tag{3.2}$$

Proof 3.1 By taking the derivative one immediately sees that it is a solution.

[2] There are numerous examples in physics. The principle of minimal substitution for a Dirac-particle in an electromagnetic field (demanding gauge-invariance, see e.g. [13]), the derivation of the Maxwell-equations by symmetry-considerations (demanding Lorentz-invariance and gauge-invariance of the action, see e.g. [1]), the Ginzburg-Landau-Hamiltonian describing the phase diagram of a magnet (demanding rotational-invariance of the action, see e.g. [14]) etc…

[3] Thus, what is meant is a Lindblad-equation relating to a fermionic many-body system with a single-particle Hilbert-space and orthonormal basis as in Notation 2.1.

Hypothesis 3.2 As physicists, we are optimistic[4] about this particular solution; we believe that we can prepare the system in such a way that exactly this solution will be realized in an experiment. Accordingly, we also hope that this solution will coincide in the limit of "long times" with an exact solution of an stationary Lindblad-equation, which we give later in eq. (4.9) for the concrete system that we study in this thesis.

Remark Due to the linearity and cyclicity of the trace, we get for the partition function

$$\frac{\mathrm{d}}{\mathrm{d}t} Z = \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Tr} \rho = \operatorname{Tr} \frac{\mathrm{d}}{\mathrm{d}t} \rho = \operatorname{Tr} \mathscr{L}[\rho] = 0,$$

therefore $Z = \text{const}$. After proper normalization of ρ one obtains

$$Z = 1.$$

Now we follow the strategy in [9] to convert the partition function of the system, by means of a Suzuki-Trotter-decomposition, into a functional integral.

Theorem 3.3 In the functional integral formalism the partition function, corresponding to the formal solution of the Lindblad-equation, assumes in continuum-notation the form

$$Z = \operatorname{Tr}\ \rho(t_\mathrm{f}) \overset{\text{Claim}}{=} \int \mathscr{D}(\psi, \overline{\psi})\, \frac{\langle \psi_{+,0} | \rho_0 | -\psi_{-,0} \rangle}{\langle \psi_{+,0} | -\psi_{-,0} \rangle} \exp\, -S[\psi, \overline{\psi}],$$

where

$$-S[\psi, \overline{\psi}] \equiv \int \mathrm{d}t \left[(\partial_t \overline{\psi}_+) \cdot \psi_+ + \overline{\psi}_- \cdot \partial_t \psi_- + \sum_\alpha 2\, L_{\alpha,+} L^\dagger_{\alpha,-} - \left((L^\dagger_\alpha L_\alpha)_+ + (L^\dagger_\alpha L_\alpha)_- \right) \right] \tag{3.3}$$

and the other symbols will be introduced and explained in the course of the text.

To prove this, we divide ("and conquer") the problem.

Lemma 3.4 With the notation $t \equiv t_\mathrm{f} \equiv t_N$, where $t_n = n\Delta t + t_\mathrm{i}$, $\Delta t = (t_\mathrm{f} - t_\mathrm{i})/N$ and $\rho(t_n) \equiv \rho_n$ one concludes for the partition function

$$Z = \operatorname{Tr}\ \rho(t_\mathrm{f}) \overset{\text{Claim}}{=} \int \mathrm{d}\psi_{\pm,N}\, \langle \psi_{+,N} | \rho_N | -\psi_{-,N} \rangle\, \mathrm{e}^{-S_N},$$

where the notations

$$-S_N = -\left(|\psi_{+,N}|^2 + |\psi_{-,N}|^2 - \overline{\psi}_{-,N} \cdot \psi_{+,N} \right),$$
$$\int \mathrm{d}\psi_{\pm,N} \equiv \int \mathrm{d}\psi_{+,N}\, \mathrm{d}\psi_{-,N} \equiv \int \mathrm{d}\overline{\psi}_{+,N}\, \mathrm{d}\psi_{+,N}\, \mathrm{d}\overline{\psi}_{-,N}\, \mathrm{d}\psi_{-,N}$$

are introduced.

[4] This optimism for this particular solutions of differential equations is not far-fetched; e.g. the separation approach which leads to the time-independent Schrödinger-equation captures only one "sort" of solutions which however can describe the real world very well, as has been confirmed by many examples (the most prominent one is probably the orbital model in atomic physics).

Proof 3.4 See Appendix, Proofs, page 63.

To prove the above theorem we need one more lemma:

Lemma 3.5 For $n \in \{0, \dots, N-1\}$ one obtains with the Lindblad-equation

$$\langle \psi_{+,n+1} | \rho_{n+1} | -\psi_{-,n+1} \rangle = \int \mathrm{d}\psi_{\pm,n} \, \langle \psi_{+,n} | \rho_n | -\psi_{-,n} \rangle \, \mathrm{e}^{-\Delta t \, S_{[n,n+1]}} + \mathcal{O}(\Delta t^2),$$

where

$$-\Delta t \, S_{[n,n+1]} = \Delta t \left(\frac{\overline{\psi}_{+,n+1} - \overline{\psi}_{+,n}}{\Delta t} \cdot \psi_{+,n} + \overline{\psi}_{-,n} \cdot \frac{\psi_{-,n+1} - \psi_{-,n}}{\Delta t} + \mathscr{L}(\overline{\psi}_{\pm,n+1}, \psi_{\pm,n}) \right),$$

and

$$\mathscr{L}(\overline{\psi}_{\pm,n+1}, \psi_{\pm,n}) = \sum_\alpha 2 \, L_\alpha(\overline{\psi}_{+,n+1}, \psi_{+,n}) L_\alpha^\dagger(-\overline{\psi}_{-,n}, -\psi_{-,n+1}) - \left((L_\alpha^\dagger L_\alpha)(\overline{\psi}_{+,n+1}, \psi_{+,n}) + (L_\alpha^\dagger L_\alpha)(-\overline{\psi}_{-,n}, -\psi_{-,n+1}) \right).$$

Proof 3.5 See Appendix, Proofs, page 64.

Proof of Theorem 3.3. We merge the intermediate results and exchange the matrix element $\langle \psi_{+,N} | \rho_N | -\psi_{-,N} \rangle$ by successively adding further auxiliary integrals according to Lemma 3.5 for a matrix element of $\rho_0 \equiv \rho(t_\mathrm{i})$:

$$\begin{aligned}
Z &= \mathrm{Tr}\ \rho(t_\mathrm{f}) \\
&\overset{\text{Lemma 3.4}}{=} \int \mathrm{d}\psi_{\pm,N} \, \langle \psi_{+,N} | \rho_N | -\psi_{-,N} \rangle \, \mathrm{e}^{-S_N} \\
&\overset{\text{Lemma 3.5}}{=} \int \mathrm{d}\psi_{\pm,N} \, \mathrm{d}\psi_{\pm,N-1} \, \langle \psi_{+,N-1} | \rho_{N-1} | -\psi_{-,N-1} \rangle \, \mathrm{e}^{-\Delta t \, S_{[N-1,N]}} \, \mathrm{e}^{-S_N} + \mathcal{O}(\Delta t^2) \\
&\overset{\text{Lemma 3.5}}{=} \int \mathrm{d}\psi_{\pm,N} \mathrm{d}\psi_{\pm,N-1} \mathrm{d}\psi_{\pm,N-2} \, \langle \psi_{+,N-2} | \rho_{N-2} | -\psi_{-,N-2} \rangle \, \mathrm{e}^{-\Delta t \, S_{[N-2,N-1]}} \mathrm{e}^{-\Delta t \, S_{[N-1,N]}} \mathrm{e}^{-S_N} + \mathcal{O}(\Delta t^2) \\
&\overset{\dots}{=} \dots \\
&\equiv \int \mathrm{d}\psi_{\pm,N} \mathrm{d}\psi_{\pm,N-1} \cdots \mathrm{d}\psi_{\pm,0} \, \frac{\langle \psi_{+,0} | \rho_0 | -\psi_{-,0} \rangle}{\langle \psi_{+,0} | -\psi_{-,0} \rangle} \, \mathrm{e}^{-\left(S_0 + \Delta t \left[S_{[0,1]} + \dots + S_{[N-1,N]} \right] + S_N \right)} + \mathcal{O}(\Delta t^2).
\end{aligned}$$

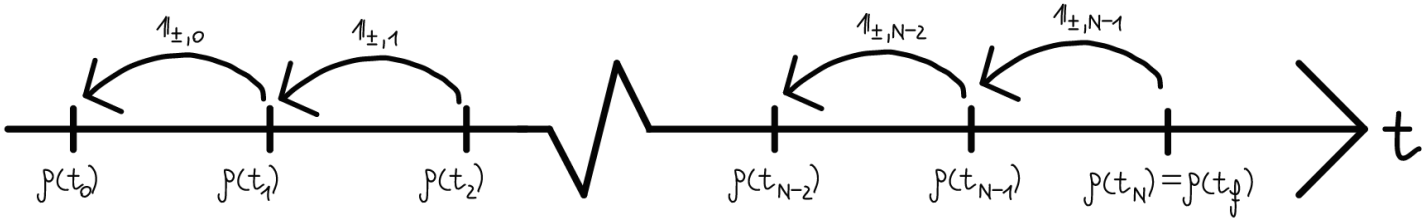


Figure 3.2: Schematic visualization for the functional integral construction; The matrix element $\langle \psi_{+,N} | \rho_N | -\psi_{-,N} \rangle$ is successively replaced with the matrix element $\langle \psi_{+,0} | \rho_0 | -\psi_{-,0} \rangle$ by inclusion of auxiliary integrals.

In the limit $N \to \infty$ we introduce the continuum-notation[5] for the action

$$\begin{aligned} -S[\psi,\overline{\psi}] = \lim_{N\to\infty} \Delta t \sum_{n=0}^{N-1} \left[\frac{\overline{\psi}_{+,n+1} - \overline{\psi}_{+,n}}{\Delta t} \cdot \psi_{+,n} + \overline{\psi}_{-,n} \cdot \frac{\psi_{-,n+1} - \psi_{-,n}}{\Delta t} \quad + \quad \mathscr{L}(\overline{\psi}_{\pm,n+1}, \psi_{\pm,n}) \right] \\ + \overline{\psi}_{-,N} \cdot \psi_{+,N} - (\overline{\psi}_{+,N} \cdot \psi_{+,N} + \overline{\psi}_{-,N} \cdot \psi_{-,N} + \overline{\psi}_{+,0} \cdot \psi_{-,0}) \\ \equiv \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \left[(\partial_t \overline{\psi}_+) \cdot \psi_+ + \overline{\psi}_- \cdot \partial_t \psi_- + \sum_\alpha 2 L_{\alpha,+} L^\dagger_{\alpha,-} - \left((L^\dagger_\alpha L_\alpha)_+ + (L^\dagger_\alpha L_\alpha)_- \right) \right], \end{aligned}$$

with $\psi_{\bullet,n} \equiv \psi_\bullet(t_n) \equiv \psi_\bullet$, etc. The continuum-notation is also introduced for the integration measure

$$\lim_{N\to\infty} \int \mathrm{d}\psi_{\pm,N} \mathrm{d}\psi_{\pm,N-1} \cdots \mathrm{d}\psi_{\pm,0} \equiv \int \mathscr{D}(\psi,\overline{\psi}),$$

and thus we obtain the desired result.

Remark If one remembers[6] that expressions like $\overline{\psi}_- \cdot \partial_t \psi_-$ are actually to be understood as scalar products then one can, referring to an orthonormal basis of a single-particle Hilbert-space, express e.g. the time-derivatives in the action as

$$\begin{aligned} \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \overline{\psi}_- \cdot \partial_t\, \psi_- &\equiv \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t\, \overline{\psi}_-(t) \cdot \partial_t\, \psi_-(t) \\ &\equiv \int_{t_\mathrm{i}}^{t_\mathrm{f}} \mathrm{d}t \sum_i \overline{\psi}_{-,i}(t)\, \partial_t\, \psi_{-,i}(t) \\ &\equiv \sum\!\!\!\!\!\!\int_{t,i} \overline{\psi}_{-,i}(t)\, \partial_t\, \psi_{-,i}(t). \end{aligned}$$

Corollary 3.6 Looking at the action in eq. (3.3) reveals that some cancellation schemes apply for certain modifications.

- For quantum jump operators L_α that contain only monomials that are *linear* in creation- and annihilation-operators it follows that equalling the indices $(+,-) \mapsto (+,+)$ truncates the quadratic sector of the action.
- For quantum jump operators L_α that contain only monomials that are *quadratic* in creation- and annihilation-operators it follows that equalling the indices $(+,-) \mapsto (+,+)$ truncates the quartic sector of the action.

Proof 3.6 See Appendix, Proofs, page 65.

Remark 3.6.1 This observation can also be regarded as a "sanity-check" in order to assess whether certain results are physically meaningful.

[5] For scalar field theories (real-valued integration variables) and bosonic systems (complex-valued integration variables) it can heuristically be explained why the transition to the integral often makes sense. (Keywords: "saddle point approximation" and / or "random phase cancellations"). For fermionic systems this notation is purely formal as there is no metric for Graßmann-numbers. Nevertheless, the charm of the continuum-notation has prevailed also for fermionic systems and this has its reasons: With this notation and the Fourier-transform, one obtains a standard-procedure with which one can simplify and approximate the formal derivatives in the action. The continuum-notation also has weaknesses, which we do not discuss further here. Words of warning about this notation can be found, for example, in [1], [5], [18], [19].

[6] Cf. Notation 2.2 or [1], p. 166-167.

Hypothesis 3.7 In the limit of "long times" the expectation values of the system should not depend on the initial state. We therefore approximate

$$\frac{\langle \psi_{+,0} | \rho_0 | - \psi_{-,0} \rangle}{\langle \psi_{+,0} | - \psi_{-,0} \rangle} \approx 1.$$

Remark 3.7.1 (without proof): In the special case

$$\rho(t=0) = \sum_i p_i | \, \eta_i \, \rangle \langle \, \eta_i \, |$$

where

$$\sum_i p_i = 1 \qquad \text{and} \qquad | \, \eta_i \, \rangle \in \mathscr{H} = \mathscr{F}(\mathcal{H}),$$

one can mathematically support the above hypothesis for the concrete system, which we consider from chapter 4 on: With the Wick-theorem one obtains expectation values which are consistent with the above approximation, except for disturbance terms which are exponentially damped.

Lindbladian Occupation Number Expectation Values as Correlators

Some standard expectation values are those that link the occupation numbers to the correlators.

Corollary 3.8 For $|N - \widetilde{N}| \leq 1$ it follows

- $\langle \psi_{+,N} \overline{\psi}_{-,N} \rangle = -\langle a^\dagger a \rangle$,
- $\langle \psi_{-,N} \overline{\psi}_{+,N} \rangle = 1 - \langle a^\dagger a \rangle$,
- $\langle \psi_{+,N} \overline{\psi}_{+,\widetilde{N}} \rangle = \mathbb{1}_{N=\widetilde{N}} \cdot \left(1 - \langle a^\dagger a \rangle\right) + \mathbb{1}_{N<\widetilde{N}} \cdot \left(-\langle a^\dagger a \rangle\right)$,
- $\langle \psi_{-,N} \overline{\psi}_{-,\widetilde{N}} \rangle = \mathbb{1}_{N=\widetilde{N}} \left(1 - \langle a^\dagger a \rangle\right) + \mathbb{1}_{N>\widetilde{N}} \cdot \left(-\langle a^\dagger a \rangle\right)$.

Proof 3.8 The crux to derive the given identities is to perform the first step in the transition to the functional integral carefully and in different ways, to use the basic properties of coherent states and the cyclicity of the trace. Thus, the proof of Lemma 3.4 serves as an important pattern to find these identities. The concrete calculations are carried out in Appendix, Proofs, page 65.

Remark 3.8.1 For actions which are quadratic in Graßmann-fields, the above expressions can be generalized for arbitrary two point functions.[7]

[7] See, for example, Theorem 5.10 or [18].

3.2 Fermionic Effective Action

3.2.1 Introduction

We consider a general[8] partition function Z in the functional integral formalism

$$Z[j,\overline{j}] = \int \mathscr{D}(\psi,\overline{\psi}) \exp\left(-S[\psi,\overline{\psi}] + \sum\!\!\!\!\!\!\int (\overline{j}\psi + \overline{\psi} j)\right), \tag{3.4}$$

with fermionic integration variables $\psi(\bullet), \overline{\psi}(\bullet)$, external Graßmann-valued sources $j(\bullet), \overline{j}(\bullet)$ and schematic notation.

Notation 3.9 We write for the (in the context of this thesis relevant) functional derivatives of a functional F with Graßmann-valued arguments (again in schematic notation)

$$\begin{aligned} F[\phi+\Delta,\overline{\phi}+\overline{\Delta}] \equiv & \quad F[\phi,\overline{\phi}] \\ & + \sum\!\!\!\!\!\!\int \left(\frac{\delta}{\delta\phi}F[\phi,\overline{\phi}], \frac{\delta}{\delta\overline{\phi}}F[\phi,\overline{\phi}]\right)\cdot\begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix} \\ & + \frac{1}{2!}\sum\!\!\!\!\!\!\int\sum\!\!\!\!\!\!\int \begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix}^{\mathrm{T}}\cdot\begin{pmatrix}\frac{\delta^2}{\delta\phi\,\delta\phi}F[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}F[\phi,\overline{\phi}]\\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}F[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}F[\phi,\overline{\phi}]\end{pmatrix}\cdot\begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix} \\ & + \mathcal{O}\left(\Delta^3, \Delta^2\overline{\Delta}^1, \Delta^1\overline{\Delta}^2, \overline{\Delta}^3\right). \end{aligned}$$

Remark 3.9.1 When it comes to functional derivatives with Graßmann-valued variables one must be very careful if one wants to keep all signs error free: One sees, for example, that in the above expansion the simple functional derivatives are to be understood as "derivatives from the right", while in the case of the double functional derivatives one derivative is to be taken "from the left", while the other is to be taken "from the right". It's also convenient to let (simple and double) functional derivatives act "from the left". This is then marked by a corresponding arrow (see below). More identities regarding these subtleties can be found in Lemma 3.20.

Notation 3.10 We write for the analog of Helmholtz free energy in the canonical ensemble[9]

$$W[j,\overline{j}] = -\log Z[j,\overline{j}] \tag{3.5}$$

and for the expectation values

$$\begin{aligned} \phi[\bullet,j,\overline{j}] &= -\frac{\vec{\delta}}{\delta\overline{j}(\bullet)}W[j,\overline{j}] = \langle\psi(\bullet)\rangle_{j,\overline{j}} = \frac{\delta}{\delta\overline{j}(\bullet)}W[j,\overline{j}], \\ \overline{\phi}[\bullet,j,\overline{j}] &= +\frac{\vec{\delta}}{\delta j(\bullet)}W[j,\overline{j}] = \langle\overline{\psi}(\bullet)\rangle_{j,\overline{j}} = -\frac{\delta}{\delta j(\bullet)}W[j,\overline{j}]. \end{aligned}$$

Remark 3.10.1 For $j,\overline{j} \neq 0$ it follows $\phi, \overline{\phi} \overset{\text{in general}}{\neq} 0$.

[8] I. e. not necessarily the one from the previous subsection

[9] In the literature [5], [11] this quantity is known as "generating functional for connected correlation functions". In the following we will simply call this quantity "free energy".

Notation 3.11 With the assumption that the mapping $(j, \bar{j}) \mapsto (\phi, \overline{\phi})$ is invertible, and with suppression of the dependencies $j \equiv j[\bullet, \phi, \overline{\phi}]$, we write the fermionic effective action (being the analog of the Gibbs-potential) as a Legendre-transformation of W

$$\Gamma[\phi, \overline{\phi}] = W[j, \bar{j}] + \int\!\!\!\!\!\!\sum (\bar{j}\phi + \overline{\phi} j). \tag{3.6}$$

Remark 3.11.1 One immediately concludes that

$$\begin{aligned} j[\bullet, \phi, \overline{\phi}] &= \frac{\vec{\delta}}{\delta \overline{\phi}(\bullet)} \Gamma[\phi, \overline{\phi}] \quad = -\frac{\delta}{\delta \overline{\phi}(\bullet)} \Gamma[\phi, \overline{\phi}], \\ \bar{j}[\bullet, \phi, \overline{\phi}] &= -\frac{\vec{\delta}}{\delta \phi(\bullet)} \Gamma[\phi, \overline{\phi}] = +\frac{\delta}{\delta \phi(\bullet)} \Gamma[\phi, \overline{\phi}]. \end{aligned}$$

Proof 3.11.1 See Appendix, Proofs, page 69.

Notation 3.12 Under the assumption that the functional Γ admits a series expansion, and after one has established an order relation[10] for the generalized indices, one writes

$$\begin{aligned} \Gamma[\phi, \overline{\phi}] &= \sum_{n,m=0}^{\infty} \int\!\!\!\!\!\!\sum_{\substack{x_{i_1} < \ldots < x_{i_n} \\ y_{j_1} < y_{j_2} < \ldots < y_{j_{m-1}} < y_{j_m}}} \gamma(x_{i_1}, \ldots, x_{i_n}, y_{j_1}, \ldots y_{j_m})\, \overline{\phi}(x_{i_1}) \cdots \overline{\phi}(x_{i_n}) \phi(y_{j_1}) \cdots \phi(y_{j_m}) \\ &= \sum_{n,m=0}^{\infty} \frac{1}{n!\, m!} \int\!\!\!\!\!\!\sum_{\substack{x_{i_1}, \ldots, x_{i_n} \\ y_{j_1}, y_{j_2}, \ldots, y_{j_{m-1}}, y_{j_m}}} \gamma(x_{i_1}, \ldots, x_{i_n}, y_{j_1}, \ldots y_{j_m})\, \overline{\phi}(x_{i_1}) \cdots \overline{\phi}(x_{i_n}) \phi(y_{j_1}) \cdots \phi(y_{j_m}) \end{aligned} \tag{3.7}$$

with complex-valued coefficients $\gamma(x_{i_1}, \ldots, x_{i_n}, y_{j_1}, \ldots y_{j_m}) \in \mathbb{C}$, such that

$$\gamma(x_{i_{\pi(1)}}, \ldots, x_{i_{\pi(n)}}, y_{j_{\sigma(1)}}, \ldots y_{j_{\sigma(m)}}) = \operatorname{sgn}(\pi) \operatorname{sgn}(\sigma)\, \gamma(x_{i_1}, \ldots, x_{i_n}, y_{j_1}, \ldots y_{j_m})$$

for $(\pi, \sigma) \in \mathrm{S}_n \times \mathrm{S}_m$.

Remark 3.12.1 Usually, double counting is avoided for discrete indices, while it is allowed for continuous indices. Also, oftentimes the prefactors $(n!m!)^{-1}$ are included in the vertex-functions. The concrete definition of the vertex-functions usually becomes clear in the respective context. Here a concrete choice of these different prefactor conventions is presented in order to be able to draw important conclusions in the following sections (which are also valid for any other choice of prefactor convention).

Remark 3.12.2 The vertex-function $\gamma(x_1, x_2, y_1, y_2)$ is also known as the *effective interaction*. It describes the true interaction between two particles in the many-body system [12].

[10] To avoid double counting.

3.2.2 Conclusions with Symmetry Transformations

In this section we will see how symmetry considerations will constrain the shape of the coefficients γ.

Definition 3.13 We call an invertible linear mapping M (with complex conjugate $\overline{M}$) between Graßmann-fields

$$\psi(x) \mapsto (M \cdot \psi)(x) \equiv \int\!\!\!\!\!\!\sum_{y} M(x,y)\,\psi(y), \qquad \overline{\psi}(x) \mapsto (\overline{M} \cdot \overline{\psi}\,)(x) \equiv \int\!\!\!\!\!\!\sum_{y} \overline{M}(x,y)\,\overline{\psi}(y)$$

a symmetry transformation if both, the integration measure and the action remain unchanged under the action of the mapping M (and $\overline{M}$):

$$\mathscr{D}(\psi,\overline{\psi}) = \mathscr{D}(M \cdot \psi, \overline{M} \cdot \overline{\psi}\,) \quad \text{and} \quad S[\psi,\overline{\psi}] = S[M \cdot \psi, \overline{M} \cdot \overline{\psi}\,]. \tag{3.8}$$

Remark 3.13.1 The set of all symmetry transformations, equipped with composition as binary-operation, is a group. For each action and integration measure one refers to this group as the "symmetry group" of the problem [12].

Lemma 3.14 For a symmetry transformation M one has

$$W[j,\bar{j}] = W[\overline{M}^{\mathrm{T}} \cdot j, M^{\mathrm{T}} \cdot \bar{j}]. \tag{3.9}$$

implying the equations

$$\begin{aligned} \phi[\bullet, \overline{M}^{\mathrm{T}} \cdot j, M^{\mathrm{T}} \cdot \bar{j}] &= (M^{-1} \cdot \phi)[\bullet, j, \bar{j}], \\ \overline{\phi}[\bullet, \overline{M}^{\mathrm{T}} \cdot j, M^{\mathrm{T}} \cdot \bar{j}] &= (\overline{M}^{-1} \cdot \overline{\phi})[\bullet, j, \bar{j}]. \end{aligned}$$

Proof 3.14 See Appendix, Proofs, page 69.

Theorem 3.15 With the above preparations one obtains for a symmetry transformation M the equation

$$\Gamma[\phi,\overline{\phi}] = \Gamma[M^{-1} \cdot \phi, \overline{M}^{-1} \cdot \overline{\phi}]. \tag{3.10}$$

Proof 3.15 See Appendix, Proofs, page 70.

Remark 3.15.1 Thus, by equation (3.10) we obtain for each symmetry transformation M (respectively its inverse M^{-1}), by comparing coefficients in eq. (3.7), restricting conditions for the vertex-functions:

$$\gamma(x_{i_1},\dots,x_{i_n},y_{j_1},\dots y_{j_m}) = \oint_{\substack{\tilde{x}_{i_1},\dots,\tilde{x}_{i_n}\\ \tilde{y}_{j_1},\tilde{y}_{j_2},\dots,\tilde{y}_{j_{m-1}},\tilde{y}_{j_m}}} \left(\prod_{l=1}^{n}\overline{M}(x_{i_l},\tilde{x}_{i_l})\right)\left(\prod_{l=1}^{m}M(y_{j_l},\tilde{y}_{j_l})\right)\gamma(\tilde{x}_{i_1},\dots,\tilde{x}_{i_n},\tilde{y}_{j_1},\dots\tilde{y}_{j_m})$$

The shape of the coefficients is therefore dictated by the symmetries of the underlying physical system. This realization will simplify concrete calculations later.

Remark 3.15.2 The above results can be memorized by the following sentence from [5]:

The effective action Γ *shares all the symmetries of the action* S.

Remark 3.15.3 Equation (3.9) gives analogous results for the free energy W.

Remark 3.15.4 The figure summarizes the results pictographically.

$$\left.\begin{array}{l} S[\psi,\overline{\psi}] = S[M\cdot\psi,\overline{M}\cdot\overline{\psi}] \\ \mathcal{D}(\psi,\overline{\psi}) = \mathcal{D}(M\cdot\psi,\overline{M}\cdot\overline{\psi}) \end{array}\right\} \Longleftrightarrow W[j,\overline{j}] = W[\overline{M}^{\mathsf{T}}\cdot j, M^{\mathsf{T}}\cdot\overline{j}] \Longleftrightarrow \Gamma[\phi,\overline{\phi}] = \Gamma[M^{-1}\cdot\phi,\overline{M}^{-1}\cdot\overline{\phi}]$$

Figure 3.3: Summary for Symmetry Transformations.

3.2.3 The One-Loop-Approximation

Approximation 3.16 In the so-called Tree-Level-Approximation, one obtains for the fermionic effective action

$$\Gamma[\phi,\overline{\phi}] \quad \approx \quad S[\phi,\overline{\phi}]. \tag{3.11}$$

Proof 3.16 See Appendix, Proofs, page 70.

Remark 3.16.1 This approximation is usually depicted with Feynman-diagrams as in the figure, where the self-energy Σ is defined by the Dyson-equation

$$G^{-1} =: G_0^{-1} + \Sigma$$

as usual[11].

[11] Regarding the self-energy, we follow the sign-convention from [7]; In [12] there is the opposite sign-convention.

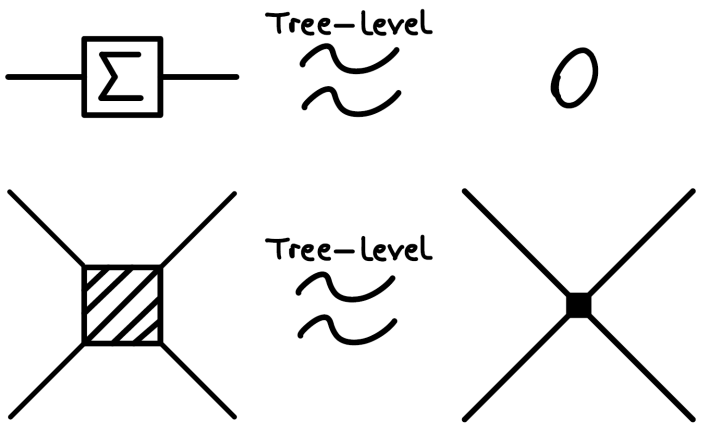


Figure 3.4: Feynman-Diagrams (schematically) for the Tree-Level-Approximation.

Corollary 3.17 Within the Tree-Level-Approximation, equations (3.6) and (3.11) can be combined, which gives

$$\frac{\delta}{\delta\phi}S[\phi,\overline{\phi}] = \overline{j} \qquad \text{and} \qquad \frac{\delta}{\delta\overline{\phi}}S[\phi,\overline{\phi}] = -j. \tag{3.12}$$

Proof 3.17 Follows immediately from the approximation and Remark 3.11.1.

Approximation 3.18 In the so-called One-Loop-Approximation, one obtains for the fermionic effective action

$$\Gamma[\phi,\overline{\phi}] \approx S[\phi,\overline{\phi}] - \frac{1}{2}\ \mathrm{Tr}\log\begin{pmatrix} \frac{\delta^2}{\delta\phi\,\delta\phi}S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}S[\phi,\overline{\phi}] \\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}S[\phi,\overline{\phi}] \end{pmatrix}, \tag{3.13}$$

where

$$\left(\frac{\delta^2}{\delta\phi\,\delta\phi}S[\phi,\overline{\phi}]\right) \equiv \left(\frac{\delta^2}{\delta\phi\,\delta\phi}S[\phi,\overline{\phi}]\right)(x,y) \equiv \frac{\delta^2}{\delta\phi(x)\,\delta\phi(y)}S[\phi,\overline{\phi}], \quad \text{etc.}$$

Proof 3.18 See Appendix, Proofs, page 72.

Notation 3.19 From now on we consider the special case[12]

$$S[\psi,\overline{\psi}] = S_0[\psi,\overline{\psi}] + S_{\mathrm{int}}[\psi,\overline{\psi}], \tag{3.14}$$

with inverse propagator $G_0^{-1} \equiv G_0^{-1}(x,y)$ in the action

[12]It is exactly this case which we will encounter in the later application; many problems of physical relevance are of this type, such as electrons in a box with periodic boundary conditions with Coulomb-interaction, or with local, attractive interaction (BCS-theory) [1].

$$S_0[\psi,\overline{\psi}] = \sum\!\!\!\!\!\!\int_{x,y} \overline{\psi}(x)\, G_0^{-1}(x,y)\, \psi(y)$$

and lastly we assume that the interaction S_{int} possesses an U(1)-invariance.

Lemma 3.20 For actions of the type in Notation 3.19 the following identities hold

$$\frac{\delta^2}{\delta\overline{\psi}(x_1)\delta\psi(x_2)} = -\frac{\vec{\delta}^2}{\delta\overline{\psi}(x_1)\delta\psi(x_2)} = -\frac{\delta^2}{\delta\psi(x_2)\delta\overline{\psi}(x_1)}. \tag{3.15}$$

Proof 3.20 See Appendix, Proofs, page 72.

Approximation 3.21 Expanding the r.h.s. of eq. (3.13) up to quartic order in the Graßmann-fields and neglecting higher order terms yields

$$\begin{aligned}
\Gamma[\phi,\overline{\phi}] \quad \approx \quad S[\phi,\overline{\phi}] \quad - \quad & \operatorname{Tr}\left[G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S_{\text{int}}[\phi,\overline{\phi}]\right] \\
- \quad \frac{1}{2} \quad & \operatorname{Tr}\left[G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S_{\text{int}}[\phi,\overline{\phi}] \cdot G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\phi} S_{\text{int}}[\phi,\overline{\phi}]\right] \\
+ \quad \frac{1}{2} \quad & \operatorname{Tr}\left[G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S_{\text{int}}[\phi,\overline{\phi}] \cdot G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S_{\text{int}}[\phi,\overline{\phi}]\right]
\end{aligned} \tag{3.16}$$

Proof 3.21 See Appendix, Proofs, page 73.

Remark 3.21.1 This approximation is usually depicted with Feynman-diagrams as in the figure. The corrections to the Tree-Level contribution of the effective interaction $\gamma(x_1,x_2,x_3,x_4)$ are the particle-particle- and particle-hole-diagrams respectively.

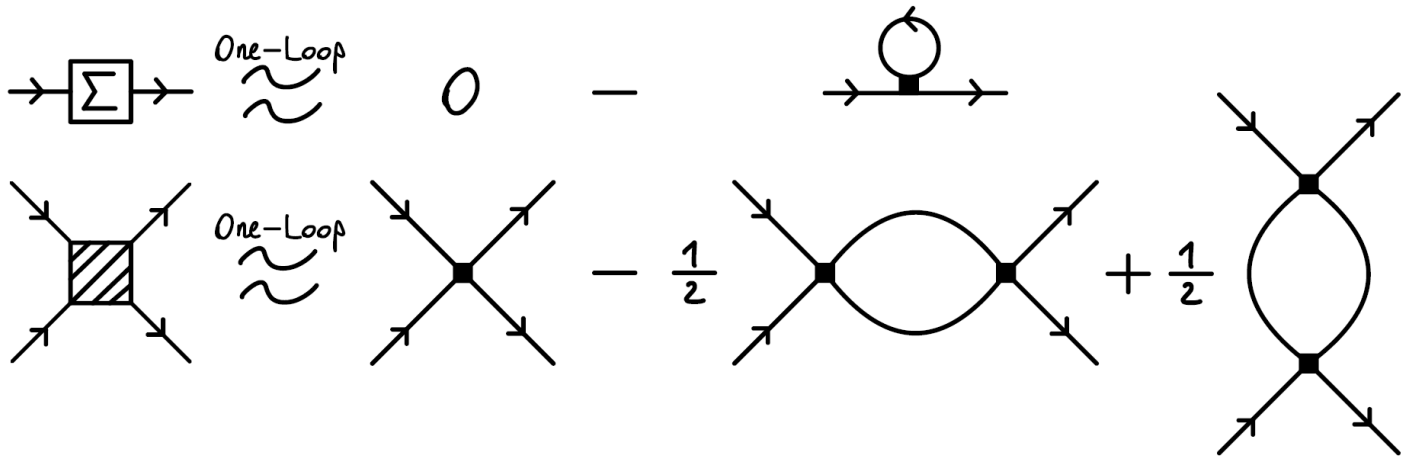


Figure 3.5: Feynman-Diagrams (schematically) for the One-Loop-Approximation.

3.3 Fermionic Functional Renormalization

3.3.1 Functional Differential Equation for the Free Energy

The starting point is the special case in Notation 3.19. First, we modify our inverse propagator in such a way that we implement a dependence on a parameter $\Lambda \in [0, \infty)$:

$$G_0^{-1}(x,y) \equiv G_0^{-1} \mapsto G_{0,\Lambda}^{-1} \equiv G_{0,\Lambda}^{-1}(x,y), \tag{3.17}$$

where we choose the Λ-dependence such that

$$G_{0,\Lambda}^{-1}(x,y) = \begin{cases} G_0^{-1}(x,y) & \text{für } \Lambda \to 0 \\ \infty & \text{für } \Lambda \to \infty \end{cases} \tag{3.18}$$

is satisfied. The philosophy of this scheme will be explained in section 3.4.1. This dependence will be inherited by all quantities of the last sections in an obvious way, which will be implemented in the notation (also in an obvious way).

Lemma 3.22 Using the so-called "source-trick" from [12], one obtains the following identity for the free energy:

$$\left(\frac{\partial}{\partial\Lambda} W_\Lambda\right)[j,\bar{j}] = \mathrm{Tr}\left[\partial_\Lambda G_{0,\Lambda}^{-1} \cdot \frac{\delta^2}{\delta\bar{j}\,\delta j} W_\Lambda[j,\bar{j}]\right] - \sum\!\!\!\!\!\!\int_{x,y} \left(\frac{\vec{\delta}}{\delta j(x)} W_\Lambda[j,\bar{j}]\right) \left(\partial_\Lambda G_{0,\Lambda}^{-1}\right)(x,y) \left(\frac{\vec{\delta}}{\delta\bar{j}(y)} W_\Lambda[j,\bar{j}]\right)$$

Proof 3.22 See Appendix, Proofs, page 75.

Remark 3.22.1 For reasons that are explained at length in the literature [7], [12], [20] this is not the flow equation that we seek to work with.

Remark 3.22.2 For later uses we mention the trivial identity

$$-\left(\frac{\partial}{\partial\Lambda} W_\Lambda\right)[j,\bar{j}] = \left(\frac{\partial_\Lambda Z_\Lambda}{Z_\Lambda}\right)[j,\bar{j}].$$

3.3.2 Functional Differential Equation for the Effective Action

This section is dedicated to the derivation of the Wetterich-equation.

Lemma 3.23 On the domain and image which guarantee the invertibility of the mapping $(j,\bar{j}) \mapsto (\phi,\bar{\phi})$ one obtains the inverse rule

$$\begin{pmatrix} -\frac{\delta^2}{\delta\bar{j}\,\delta j} W_\Lambda[j,\bar{j}] & \frac{\delta^2}{\delta\bar{j}\,\delta\bar{j}} W_\Lambda[j,\bar{j}] \\ \frac{\delta^2}{\delta j\,\delta j} W_\Lambda[j,\bar{j}] & -\frac{\delta^2}{\delta j\,\delta\bar{j}} W_\Lambda[j,\bar{j}] \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\delta^2}{\delta\bar{\phi}\,\delta\phi} \Gamma_\Lambda[\phi,\bar{\phi}] & \frac{\delta^2}{\delta\bar{\phi}\,\delta\bar{\phi}} \Gamma_\Lambda[\phi,\bar{\phi}] \\ \frac{\delta^2}{\delta\phi\,\delta\phi} \Gamma_\Lambda[\phi,\bar{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\bar{\phi}} \Gamma_\Lambda[\phi,\bar{\phi}] \end{pmatrix}.$$

Proof 3.23 This identity follows immediately from the chain rule, the equations in Notation 3.10 and Remark 3.11.1. The concrete computations are carried out in Appendix, Proofs, page 76.

Theorem 3.24 The Wetterich-equation for the fermionic effective action reads

$$\partial_\Lambda \Gamma_\Lambda[\phi,\overline{\phi}] \quad = \quad -\frac{1}{2}\,\mathrm{Tr}\left(\frac{\begin{pmatrix}\partial_\Lambda G_{0,\Lambda}^{-1} & 0\\ 0 & -\partial_\Lambda G_{0,\Lambda}^{-\mathrm{T}}\end{pmatrix}}{\begin{pmatrix}\frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_\Lambda[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}\Gamma_\Lambda[\phi,\overline{\phi}]\\ \frac{\delta^2}{\delta\phi\,\delta\phi}\Gamma_\Lambda[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}\Gamma_\Lambda[\phi,\overline{\phi}]\end{pmatrix}}\right) \quad + \quad \sum\!\!\!\!\!\!\int_{x,y} \overline{\phi}(x)\,\partial_\Lambda G_{0,\Lambda}^{-1}(x,y)\,\phi(y). \tag{3.19}$$

Proof 3.24 See Appendix, Proofs, page 77.

Remark 3.24.1 The quotient of two operators is well-defined here, although we do not necessarily have commuting operators, because the trace is cyclic.

Remark 3.24.2 In [6], it is shown how this equation is equivalent to the standard form of the Wetterich-equation from [21], by introducing the so-called "Wetterich-Legendre-Transformation" $\Gamma_\Lambda^{\mathrm{We}}$ [12].

Notation 3.25 For notational simplicity we introduce the following abbreviations

$$\begin{aligned}
&\bullet\ \boldsymbol{\delta}^2\Gamma_\Lambda[\phi,\overline{\phi}] = \begin{pmatrix}\frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_\Lambda[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}\Gamma_\Lambda[\phi,\overline{\phi}]\\ \frac{\delta^2}{\delta\phi\,\delta\phi}\Gamma_\Lambda[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}\Gamma_\Lambda[\phi,\overline{\phi}]\end{pmatrix},\\
&\bullet\ \boldsymbol{G}_\Lambda^{-1} \quad = \boldsymbol{\delta}^2\Gamma_\Lambda[0,0],\\
&\bullet\ \boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}] \quad = \boldsymbol{\delta}^2\Gamma_\Lambda[0,0] - \boldsymbol{\delta}^2\Gamma_\Lambda[\phi,\overline{\phi}],\\
&\bullet\ \boldsymbol{G}_{0,\Lambda}^{-1} \quad = \begin{pmatrix}\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1} & 0\\ 0 & -\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-\mathrm{T}}\end{pmatrix},\\
&\bullet\ \boldsymbol{S}_\Lambda(x,y) \quad = \left(\boldsymbol{G}_\Lambda \cdot \partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1} \cdot \boldsymbol{G}_\Lambda\right)(x,y),\\
&\bullet\ \boldsymbol{S}_\Lambda \quad = \begin{pmatrix}S_\Lambda & 0\\ 0 & -S_\Lambda^{\mathrm{T}}\end{pmatrix}.
\end{aligned}$$

Remark 3.25.1 The quantity $\boldsymbol{S}_\Lambda$ is also known as "single-scale propagator" [12]. In Remark 3.32.1 it will become clear why.

Theorem 3.26 With the above abbreviations, the fermionic Wetterich-equation can be further simplified to

$$\begin{aligned}
\partial_\Lambda \Gamma_\Lambda[\phi, \overline{\phi}] \quad &= \quad - \operatorname{Tr}\left(G_\Lambda^{-1} \cdot \partial_\Lambda G_{0,\Lambda}^{-1} \right) \\
&- \quad \frac{1}{2} \operatorname{Tr}\left(\boldsymbol{S}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda \right) \quad + \quad \sum\!\!\!\!\!\!\int_{x,y} \overline{\phi}(x)\, \partial_\Lambda G_{0,\Lambda}^{-1}(x,y)\, \phi(y) \\
&- \quad \frac{1}{2} \operatorname{Tr}\left(\boldsymbol{S}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda \cdot \boldsymbol{G}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda \right) \\
&- \quad \frac{1}{2} \operatorname{Tr}\left(\boldsymbol{S}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda \cdot \boldsymbol{G}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda \cdot \boldsymbol{G}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda \right) \\
&- \quad \dots
\end{aligned} \tag{3.20}$$

Proof 3.26 See Appendix, Proofs, page 78.

Remark 3.26.1 At this point we notice why the sign convention of $\boldsymbol{\mathcal{V}}_\Lambda$ in Notation 3.25 is reasonable, although defying the convention: In the Wetterich-equation, this gives us an infinite sum *without* alternating sign.

Remark 3.26.2 In the next section we will express the individual terms by Feynman-diagrams.

The above equation represents an infinite system of coupled integro-differential equations. As will be explained in section 3.4.1, we are particularly interested in the vertex-functions corresponding to the two-point- and four-point-correlators. A common approach is the following simplificiation.

Approximation 3.27 In a level-2 truncation[13] one neglects all terms above the quadratic order in the interaction $\boldsymbol{\mathcal{V}}_\Lambda$. Thus one arrives at the approximation

$$\begin{aligned}
\partial_\Lambda \Gamma_\Lambda[\phi, \overline{\phi}] \quad &\approx \quad - \operatorname{Tr}\left(G_\Lambda^{-1} \cdot \partial_\Lambda G_{0,\Lambda}^{-1} \right) \\
&- \quad \frac{1}{2} \operatorname{Tr}\left(\boldsymbol{S}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda \right) \quad + \quad \sum\!\!\!\!\!\!\int_{x,y} \overline{\phi}(x)\, \partial_\Lambda G_{0,\Lambda}^{-1}(x,y)\, \phi(y) \\
&- \quad \frac{1}{2} \operatorname{Tr}\left(\boldsymbol{S}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda \cdot \boldsymbol{G}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda \right).
\end{aligned} \tag{3.21}$$

Fact 3.28 To apply this formalism now to concrete problems[14] we need two things:

1. A concrete shape of the regulator in eq. (3.17).
2. An approximation / ansatz[15] to the effective action Γ_Λ.

Remark 3.28.1 The most general ansatz for the effective action Γ_Λ is obtained by eq. (3.7). At least for non-trivial systems, it is obvious that this approach inevitably leads to a giant combinatorial puzzle. It is also said in [7], p. 10 (with references to this thesis [...] inserted for better understanding):

[13] More generally, for $n \in \mathbb{N}_{>0}$ one can introduce a level-n truncation [6].

[14] I.e. actions of the shape in eq. (3.14)

[15] There are at least two common ansatzes for the effective action; The first one is a so-called "derivative expansion" [12], also known as "gradient expansion" [10] whereas the other common ansatz (which we will focus on in this thesis) is a so-called "vertex expansion" [12].

> *"Unfortunately, the direct solution of the functional flow equation* [eq. (3.21)] *is only possible for a very small number of systems. However, in practice, one is primarily interested in a few number of 1PI vertex functions. It is therefore convenient to expand both sides of* [eq. (3.21)] *in powers of the fields, and calculate the flow only for certain coefficients."*

To reduce the number of vertex functions in the most general ansatz in eq. (3.7) to a "small number", one usually exploits the fact that the effective action has the same symmetries as the action itself[16]. One can then do the followings things:

- One can either stare at the action of the concrete problem and conclude by symmetry considerations, analogous to the calculations in [7], to what extent the shape of the vertices can be restricted,
- or one can start from a Tree-Level- or One-Loop-Approximation to make an ansatz for the effective action.

The advantage of the latter procedure, which we will use in this thesis, is that one does not have to bother with the details of symmetry transformations and the underlying Lie-groups and Lie-algebras: The non-vanishing vertex-functions in the Approximation(s) 3.16, 3.18 satisfy the required symmetry properties *by construction*[17].

Another common strategy, that we will also make use of, is to classify the appearing terms on the r.h.s. of the flow equations (3.21) by their magnitude [6].

Hypothesis 3.29 *The only* vertex-functions that govern the macroscopic behaviour of the system[18], are those, that flow under the influence of terms, that diverge in the limit $\Lambda \longrightarrow 0$ (assuming their existence).

Remark 3.29.1 Building on the analogy to Wilson's RG-procedure[19] one also realizes, that diverging terms not only govern the contributions to the effective action (with decreasing Λ) but as one follows the flow, most of the other terms also drop out due to their "scaling dimension"[20]! Thus it is also said in [12], p. 78-79

> *"Microscopic lattice models are usually characterized by infinitely many coupling constants with negative scaling dimensions, implying the existence of infinitely many irrelevant scaling variables. (...) As the RG is iterated there is an enormous reduction of the dimensionality of the coupling space and all irrelevant terms drop out. The leading singular behavior of physical observables is controlled by a small number of relevant (and marginally relevant) couplings."*

where we will introduce the word "marginal" in section 3.3.4.

[16] Cf. Remark 3.15.2

[17] The proof of this statement for the ansatz by the One-Loop-Approximation is not carried out in this thesis; We content ourselves, for reasons that will become clear later, with an approach by the Tree-Level-Approximation which, according to Theorem 3.15 satisfies all the required symmetry properties.

[18] As will be elaborated in section 3.4.1: The vertex-functions correspond to correlators, which are known to form the bridge between theory and experiment.

[19] This will be elaborated in section 3.4.2.

[20] This will be introduced in Chapter 6.

3.3.3 An Example for Flow Equations

Theorem 3.30 One can establish a correspondence between eq. (3.16) and eq. (3.21) if one wisely chooses the regulator and the approximation / ansatz for the effective action.

The proof of this theorem is embedded in the text of this subsection, and ends with Remark 3.33.1.

Choice 1: Regulator

In eq. (3.17) we choose

$$G^{-1}_{0,\Lambda}(x,y) := \left(\mathrm{Mat}[\theta_{\Lambda,\epsilon}] \cdot G_0\right)^{-1}(x,y)$$

where $\mathrm{Mat}[\theta_{\Lambda,\epsilon}](x,y)$ denotes a diagonal, regularized[21] Heaviside-matrix

$$\mathrm{Mat}[\theta_{\Lambda,\epsilon}](x,y) := \delta(x-y)\,\theta_\epsilon\left(f(x)-\Lambda\right), \tag{3.22}$$

with ...

- $\delta(x-y)$ being a product of a Kronecker-delta and a Dirac-delta for the discrete and continuous part of the generalized indices,
- a "usual" regularized Heaviside-function $\theta_\epsilon(\bullet-\Lambda) \equiv \theta_{\Lambda,\epsilon}(\bullet)$ with step-position Λ,
- a regularization-parameter $\epsilon > 0$,
- a system specific, bounded function $f(x) > 0$.

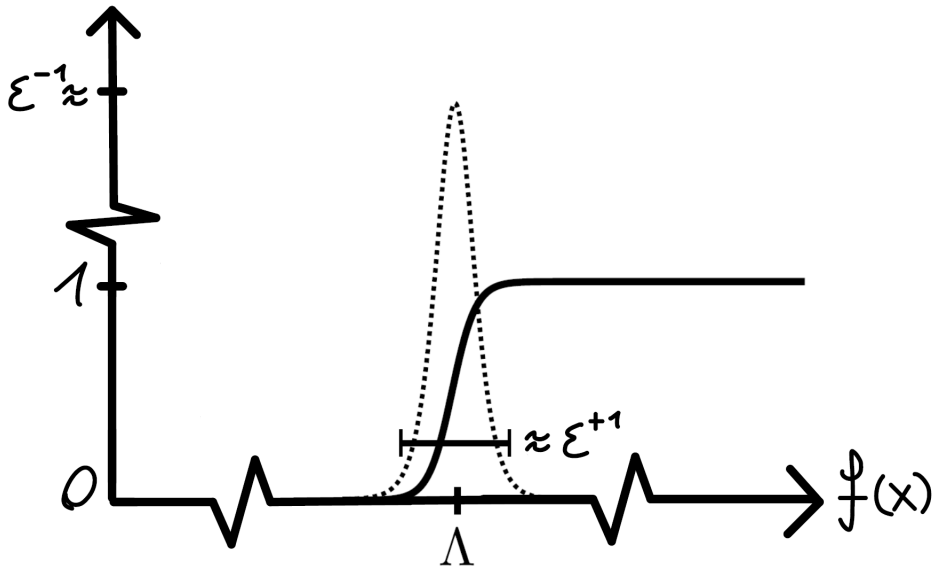


Figure 3.6: Schematically: The graph of the functions
$f(x) \mapsto \theta_\epsilon(f(x)-\Lambda)$,
$f(x) \mapsto |\partial_\Lambda \theta_\epsilon(f(x)-\Lambda)|$

[21] A sharp cutoff sometimes leads to technical complications, like performing an integral involving a Heaviside- *and* a Dirac-function. It is a prime example of non-commuting limits in physics. However, a regularized step-function is in many cases still much easier to work with than other choices. For notational simplicity it is also usual to neglect the ϵ-dependence for most of the time and to just bring it up, when one needs it [12].

Choice 2: Approximation / Ansatz for the Effective Action

For the effective action we choose the following ansatz

$$\Gamma_\Lambda[\phi,\overline{\phi}] \overset{\text{Ansatz}}{\approx} c(\Lambda) + \Gamma_{0,\Lambda}[\phi,\overline{\phi}] + \Gamma_{\text{int},\Lambda}[\phi,\overline{\phi}],$$

where ...

- $c(\Lambda) = -\int \mathrm{d}\Lambda \ \mathrm{Tr}\left(G_\Lambda^{-1}\cdot\partial_\Lambda G_{0,\Lambda}^{-1}\right)$ is a formal constant[22],
- $\Gamma_{0,\Lambda}[\phi,\overline{\phi}] = \sum\!\!\!\!\!\!\int_{x,y} \overline{\phi}(x)\, G_{0,\Lambda}^{-1}(x,y)\,\phi(y)$,
- $\Gamma_{\text{int},\Lambda}[\phi,\overline{\phi}] = \sum\!\!\!\!\!\!\int_{(z_1,\dots,z_4)\in I} \gamma^\Lambda(z_1,\dots,z_4)\,\overline{\phi}(z_1)\overline{\phi}(z_2)\phi(z_3)\phi(z_4)$,
- $I = \left\{(z_1,\dots,z_4) \mid \gamma_{\text{Tree}}(z_1,\dots,z_4)\neq 0\right\}$

and initial conditions

$$\Gamma_{\text{int},\infty}[\phi,\overline{\phi}] = S_{\text{int}}[\phi,\overline{\phi}]. \tag{3.23}$$

Remark 3.30.1 This is a similar ansatz as in [6], page 41: The self-energy-feedback is being neglected, but on the other hand we keep the structure of the interaction from the bare action, and *include* a Λ-dependence of the vertex-functions. A similar strategy was used in [12] p. 229-232 and [10], p. 312.

Corollary 3.31 These two choices satisfy all the requirements demanded by Theorem 3.15 and eq. (3.18).

Proof 3.31 See Appendix, Proofs, page 79.

Corollary 3.32 With the above choices one obtains the identities

$$\begin{aligned}
&\bullet\ G_{0,\Lambda}(x,y) = \left(\mathrm{Mat}[\theta_{\Lambda,\epsilon}]\cdot G_0\right)(x,y),\\
&\bullet\ S_\Lambda(x,y) \approx \left(\mathrm{Mat}[\delta_{\Lambda,\epsilon}]\cdot G_0\right)(x,y) = -\partial_\Lambda G_{0,\Lambda}(x,y),\\
&\bullet\ \boldsymbol{G}_\Lambda^{-1} \approx \boldsymbol{G}_{0,\Lambda}^{-1}.
\end{aligned}$$

where $\delta_{\Lambda,\epsilon}(\bullet) \equiv \delta_\epsilon(\bullet-\Lambda) \equiv -\partial_\Lambda\theta_\epsilon(\bullet-\Lambda)$ denotes a regularized Dirac-delta.

Proof 3.32 See Appendix, Proofs, page 80.

Remark 3.32.1 With the appearance of the regularized Dirac-delta $\delta_{\Lambda,\epsilon}$ in S_Λ one immediately grasps the name "single-scale propagator" introduced in remark 3.25.1.

[22] The term $c(\Lambda)$ corresponds to the term in eq. (3.21) that is constant in the fields.

Theorem 3.33 For the third summand on the r.h.s. in the simplified flow equation (eq. (3.21)) one has

$$\begin{aligned} -\frac{1}{2}\operatorname{Tr}\left(\boldsymbol{S}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda \cdot \boldsymbol{G}_\Lambda \cdot \boldsymbol{\mathcal{V}}_\Lambda\right) \approx \quad & -\operatorname{Tr}\left[\partial_\Lambda G_{0,\Lambda} \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}\Gamma_{\mathrm{int},\Lambda}[\phi,\overline{\phi}] \cdot G_{0,\Lambda}^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\phi}\Gamma_{\mathrm{int},\Lambda}[\phi,\overline{\phi}]\right] \\ & +\operatorname{Tr}\left[\partial_\Lambda G_{0,\Lambda} \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_{\mathrm{int},\Lambda}[\phi,\overline{\phi}] \cdot G_{0,\Lambda} \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_{\mathrm{int},\Lambda}[\phi,\overline{\phi}]\right] \end{aligned} \tag{3.24}$$

Proof 3.33 See Appendix, Proofs, page 80.

Remark 3.33.1 By direct comparison with eq. (3.16) one sees the striking correspondence to contributions of the effective interaction in the One-Loop-Approximation. This also becomes clear, when we draw the corresponding Feynman-diagrams, denoting the derivative of a propagator by a slashed line: One immediately recognizes the typical particle-particle- and particle-hole-diagrams.

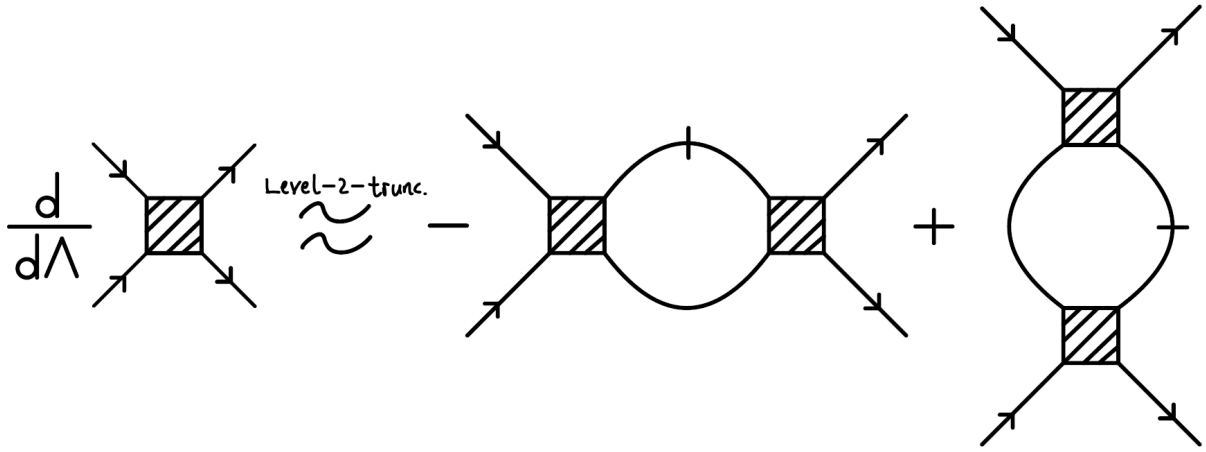


Figure 3.7: Feynman-Diagrams (schematically) for the flow-equations within the level-2-truncation with the above choices for regulator and effective action.

Remark 3.33.2 In [5] it is also pointed out, how one can reproduce from this equation the One-Loop-Approximation by approximating the effective interaction and integrating the resulting equation. This also recovers the factors of $1/2$ in eq. (3.16) and the diagrammatic language of Remark 3.21.1: Precisely this factor is produced by integrating the expressions of the type "$\partial_\Lambda G_\Lambda \cdots G_\Lambda \cdots$" to expressions of the type "$G_\Lambda \cdots G_\Lambda \cdots$" and using properties like cyclicity of the trace, invariance under transposition etc.

Remark 3.33.3 It is reasonable to note that we used different sets of identities to arrive at the respective equations: For the One-Loop-approximation in eq. (3.16), we performed a trace-log-expansion up to second order in the interaction. For the level-2-truncation leading to eq. (3.24) we used the Neumann-series and truncated the expansion after the second order in the interaction. Different techniques yielding equivalent sets of equations obviously indicate their correctness.

3.3.4 Universality

Canonical Scaling Analysis

In order to introduce critical exponents, which are quantities of interest when it comes to phase transitions, one has to introduce dimension*less* vertex-functions, as elaborated in [6], [8] and [12].

Fact 3.34 Since the action in notation 3.19

$$S[\psi,\overline{\psi}] = \sum\!\!\!\!\!\!\int_{x,y} \overline{\psi}(x)\, G_0^{-1}(x,y)\, \psi(y) + S_{\text{int}}[\psi,\overline{\psi}]$$

must be dimensionless[23], it is evident that the fields ψ,$\overline{\psi}$ and the inverse propagator G_0^{-1} possess a dimension that compensate dimensions from the possible integration measures in the action. Therefore, with Notation 3.10 and the series expansion in eq. (3.7) it is evident[24] that the vertex functions

$$\gamma(x_{i_1},\dots,x_{i_n},y_{j_1},\dots y_{j_m})$$

and fields ϕ, $\overline{\phi}$ must carry a dimension.

Remark 3.34.1 The common way of introducing dimensionless vertex-functions is by dividing the vertex-functions from the series expansion in eq. (3.7) by a suitable power of Λ from eq. (3.22).

Remark 3.34.2 We will elaborate this "rescaling"-process for the specific problem at hand in section 6.2.1. A general treatment would go beyond the scope of this thesis.

β-Functions and Critical Exponents

Notation 3.35 We choose an enumeration[25] of the dimensionless vertex-functions in eq. (3.7) and notate them with the common symbol $\{g_1, g_2, g_3, \dots\}$ as "couplings", suppressing the Λ-dependence $g_\bullet \equiv g_\bullet(\Lambda)$.

Remark 3.35.1 Referring to the vertex-functions as couplings happens in close analogy to Wilson's RG-scheme, which will be established later in section 3.4.2.

Remark 3.35.2 The couplings are classified as "relevant", "marginal" and "irrelevant" if their magnitude increases, stays fixed or decreases respectively, upon lowering the scale Λ [10]. One of these cases will happen for sure since we divide the vertex-functions by a suitable power of Λ as outlined above, giving at least one flowing contribution from the canonical dimension of the coupling, also called *engineering dimension* [12].

[23] This simply follows from the fact that the partition function in eq. (3.4) otherwise is not a number.
[24] Since the effective action is the Legendre-transformation of the free energy, which is (in our prefactors-convention) simply a number.
[25] For continuous indices, we pick a suitable discretization as in [23].

Definition 3.36 We define the β-functions

$$\beta_1, \beta_2, \beta_3, \dots$$

from the template [8] with the flow equations as

$$\beta_i(g_1, g_2, g_3, \dots) := \partial_\Lambda g_i$$

where the r. h. s. is to be understood with the flow-eq. (3.21) and the equation relating dimensionless and dimensional couplings to each other.

Remark 3.36.1 Thus, the β-functions are vector fields that formalize the functional renormalization group flow.

Definition 3.37 By a fixed point of the renormalization group flow we mean a configuration $(g_1^*, g_2^*, g_3^*, \dots)$ such that for all $i \in \{1, 2, 3, \dots\}$ the equation

$$\beta_i(g_1^*, g_2^*, g_3^*, \dots) = 0$$

is satisfied.

Remark 3.37.1 One calls a fixed point $(g_1^*, g_2^*, g_3^*, \dots)$ *Gaussian* if

$$\sum_i |g_i^*| = 0,$$

and *non-Gaussian* if

$$\sum_i |g_i^*| > 0.$$

After one has found a fixed point, one considers the linearized flow in the vicinity of the fixed point. Introducing the abbreviations $\vec{g} \equiv (g_1, g_2, g_3, \dots)$, etc. then gives

$$\beta_i(\vec{g} - \vec{g}^*) = \sum_j \left(\frac{\partial \beta_i}{\partial g_j} \right) (\vec{g}^*) \cdot (\vec{g}_j - \vec{g}_j^*) + \mathcal{O}(\vec{g}_\bullet - \vec{g}_\bullet^*)^2$$

Notation 3.38 The matrix $B \equiv B_{i,j}$ defined by

$$B_{i,j} := \left(\frac{\partial \beta_i}{\partial g_j} \right) (\vec{g}^*)$$

is called *stability-matrix*. It's eigenvalues are the *critical exponents*.

Remark 3.38.1 Quoting from [12], p. 11 emphasizes the importance of these quantities:

> *"It turns out that the numerical values of the critical exponents are not simply given by rational numbers as one might have expected (...) Moreover, the exponents are universal in the sense that entire classes of materials consisting of very different microscopic constituents can have the same exponents. (...) All materials can thus be divided into universality classes, which are characterized by the same critical exponents."*

and also noted on p. 75 of the same book

> *"One of the most fundamental insights provided by the RG is the connection between critical exponents and the local behavior of the RG flow in the vicinity of a given RG fixed point."*

Further, to summarize our mathematical tools in this context, the functional renormalization group provides a systematic method for explicitly calculating critical exponents and therefore determining universality classes.

3.4 Comments

3.4.1 On the Scheme

As physicists, we are interested in making predictions that can be verified experimentally. As pointed out in [1] correlation functions, in particular two-point- and four-point-correlators, form the bridge between theory and experiment. For actions involving polynomial expressions in the fields with cubic or higher order these correlation functions usually cannot be calculated in closed form.

But just as in classical mechanics where there are several possibilities[26] to determine trajectories, there are also several possibilities to calculate correlation functions for non-trivial systems in quantum field theory (within certain approximations):

- If one works with the partition function in eq. (3.4), then one can approximate the correlation functions by expanding the integrand within the functional integral in its interaction.[27]
- If one works with the free energy in eq. (3.5), then one can consider derivatives of it to determine physical observables.[28]
- If one works with the effective action in eq. (3.6), then one can arrive at approximated expressions for the vertex-functions by the One-Loop-Approximation.[29]

After we used the Wetterich-equation (3.20) to replace the problem

> *"Solving non-Gaußian functional integrals."*

by the problem

> *"Solving an infinitely large system of functional integro-differential equations."*

we have other possibilities (i.e. approximation schemes). For instance the choices in section 3.3.3 are just one of a lot of other choices, leading to other flow equations which might describe the physics of the problem properly.

[26] E.g. The Newtonian force-ansatz, Lagrangian-formalism, Hamiltonian-formalism, Hamilton-Jacobi-formalism.

[27] This procedure was used, for instance in [1] at the example of the Ising-model.

[28] This was done, for instance, in [1] at the example of an electron-gas with Coulomb-interaction (in the grand canonical ensemble) to determine the expectation value of the particle number within the so-called "Random-Phase-Approximation" of the free energy.

[29] It is a standard fact (see e.g. [11]) that the correlation-functions can be determined with knowledge of the vertex-functions. This approximation was used, for example in [5] on the example of the weakly, locally and attractively interacting Bose-gas (in the grand canonical ensemble).

3.4.2 On (Functional) Renormalization

The term "renormalization" refers to several similar but in detail mostly different techniques that are typically applied to systems performing a second order phase transition. The idea is to map the original, "non-renormalized" system according to a well-defined rule to a so-called "renormalized" system by changing the scale. Oftentimes the renormalized system is simpler then.

However, in [14], p. 45 it is also said

> *"Yet the method, known as the Renormalization Group, is not exact nor completely controlled. Instead, it should be regarded as largely conceptual - i.e. its application, which relies fundamentally only on scaling, can be tailored to the particular application at hand."*

The functional renormalization (also known as FRG) that we are concerned with in this thesis corresponds to one branch of RG-techniques, which are analogous to Wilson's successive mode-elimination scheme [12].

More precisely, in Wilson's RG-procedure (exemplified, for instance, by the Ginzburg-Landau-action as one encounters it in [5]) in each RG-step the coupling constants[30] are mapped to new, renormalized coupling constants. The effect is scale invariant under a RG-step if one has found a fixed point of the couplings. With such a fixed point neither couplings, and therefore, nor correlators change under a RG-step. In particular two-point and four-point-correlators do not change. These two-point- and four-point-correlators correspond to the numerical values of the vertex-functions in the series expansion of the effective action in eq. (3.7).

Thus, if we introduce a discretization for the interval

$$[0,\infty) = \bigcup_{j=0}^{\infty} [\Lambda_j, \Lambda_{j+1}),$$

then we obtain a discretized form of the Wetterich-equation

$$\begin{aligned} \Delta_\Lambda \Gamma_\Lambda[\phi,\overline{\phi}] \approx \quad & -\frac{1}{2}\,\mathrm{Tr}\left(\frac{\Delta_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1}}{\boldsymbol{\delta}^2\Gamma_\Lambda[\phi,\overline{\phi}]}\right) + \sum\!\!\!\!\!\!\int_{x,y} \overline{\phi}(x)\,\Delta_\Lambda G_{0,\Lambda}^{-1}(x,y)\,\phi(y) \\ \Longleftrightarrow \Gamma_{\Lambda_{j+1}} \approx \quad & \Gamma_{\Lambda_j} - \frac{1}{2}\,\mathrm{Tr}\left(\frac{\boldsymbol{G}_{0,\Lambda_{j+1}}^{-1} - \boldsymbol{G}_{0,\Lambda_j}^{-1}}{\boldsymbol{\delta}^2\Gamma_{\Lambda_j}[\phi,\overline{\phi}]}\right) + \sum\!\!\!\!\!\!\int_{x,y} \overline{\phi}(x)\,\left(G_{0,\Lambda_{j+1}}^{-1} - G_{0,\Lambda_j}^{-1}\right)(x,y)\,\phi(y) \end{aligned}$$

and we get again "small" FRG-steps for the vertex-functions (which is analogous to Wilson's procedure with the above considerations).[31]

[30] There, the couplings correspond to the prefactors of all the field-terms in the action.

[31] Building on this analogy for actions that are expressed in momentum space: A specific value of $\Lambda \in [0,\infty)$ represents the boundary between short-wavelength ($|\vec{k}| \gtrsim \Lambda$) fluctuations which have already been eliminated in the Wilsonian RG and the long-wavelength ($|\vec{k}| \lesssim \Lambda$) fluctuations [12].

3.4.3 On Approximations

We consider a concrete observable, that we would like to describe theoretically by considering a linear combinations of correlators as explained in [1]. Then an approximation, expressed by the symbol

$$\text{“} \approx \text{”}$$

is, in my opinion, to be understood in the following way:

> The theoretical prediction is in *some* sense, which must be specified for *each* observable and *each* system *in each concrete experimental arrangement*, “close” to the measured quantity.

Therefore an approximation can already be considered as “valid” or “good” if simply some proportionality is preserved.[32]

In this spirit, the validity of practically *any* approximation must be assessed by experiments!

[32] Exactly in this way, e. g., the approximations in the derivation of Fermi’s golden rule are to be understood if one seeks to describe with this the momentum distribution of electrons from the β^- decay measured by a scintillation counter as in [24]. The transition rate (as a function of the momentum) represents an observable. The approximations, which are used in the context of the derivation, deliver the (experimentally verified) proportionality to the density of states.

One also observes (see, for example, [25]) that treating the electrons of a copper sample as a free electron gas correctly predicts the proportionality of the electrical resistivities temperature-dependence, being the observable to be measured. Nevertheless, the temperature-independence predicted by the Wiedemann-Franz-law and the numerical value of the Lorenz-number is not reproduced for arbitrary temperatures. In this spirit, the free electron gas-model can be considered as a “good” approximation for the one observable, and as a “bad” approximation for the other observable.

4 The Lindblad-Equation of a Concrete System

The aim of this chapter is to introduce the system and therefore the Lindblad-equation from the template [15], which is what we would like to investigate in the remainder of the thesis.

4.1 A One-Dimensional Crystal with a Topological Phase Transition

4.1.1 Starting Situation

Notation 4.1 We consider a one-dimensional lattice of length $L = M \cdot a$, with periodic boundary conditions and with two[1] Wannier-states per lattice site and the corresponding fermionic creation- and annihilation-operators[2]

$$\{a_{\nu,j}, a_{\nu',j'}\} = \{a^\dagger_{\nu,j}, a^\dagger_{\nu',j'}\} = 0,$$
$$\{a_{\nu,j}, a^\dagger_{\nu',j'}\} = \delta_{\nu,\nu'}\delta_{j,j'},$$

where $(\nu, j) \in \{1,2\} \times \{1,\dots,M\}$, etc.

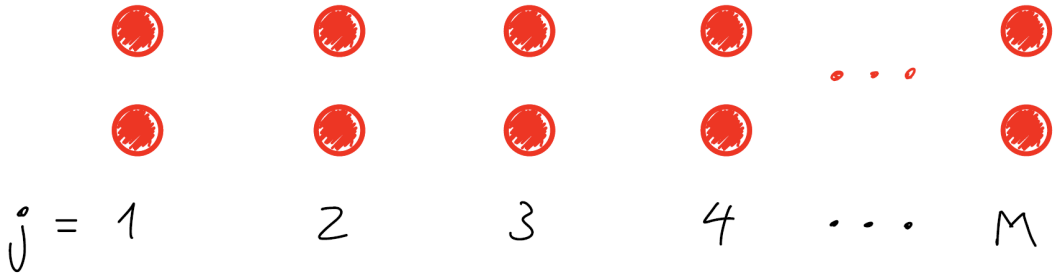


Figure 4.1: Typical pictographic representation of the initial situation.

Definition 4.2 For reasons that will become clear in the next subsection, we define[3] new operators:

$$l_{\nu,j} := \frac{1}{\sqrt{2}}\left((-1)^{\nu+1}a_{1,j+1} + a_{2,j}\right), \qquad r_{\nu,j} := \frac{1}{\sqrt{2}}\left((-1)^{\nu+1}a_{1,j} + a_{2,j+1}\right). \tag{4.1}$$

[1] These two states could correspond to the two spins of an electron, for example, or to two different energy bands.
[2] It will be clear from the context whether the symbol a denotes the lattice constant or an annihilation-operator.
[3] These operators were introduced in a different ("neater") way in the template [15]. The proof that both representations are equivalent is given in Appendix, Proofs, page 81. In order to work as "problem-oriented" as possible and to come quickly to concrete calculations and results, we stick to this representation.

Theorem 4.3 The newly introduced operators each fulfill the CAR, i.e.

$$\{l_{\nu,j}, l^\dagger_{\nu',j'}\} = \delta_{\nu,\nu'}\delta_{j,j'}, \qquad \{r_{\nu,j}, r^\dagger_{\nu',j'}\} = \delta_{\nu,\nu'}\delta_{j,j'},$$
$$\{l_{\nu,j}, l_{\nu',j'}\} = \{l^\dagger_{\nu,j}, l^\dagger_{\nu',j'}\} = 0, \qquad \{r_{\nu,j}, r_{\nu',j'}\} = \{r^\dagger_{\nu,j}, r^\dagger_{\nu',j'}\} = 0.$$

Proof 4.3 See Appendix, Proofs, page 83.

We quote (with adapted notation) from the template [15], p. 9:

> „*To get some intuition for the operators l, consider the M particle configuration defined by occupation of all l_2 states: $|L\rangle \equiv \prod_j l^\dagger_{2,j}|\,vac.\,\rangle$. Since we have $2M$ sites in total, this is a half filled state. With $l^\dagger_{2,j} = \frac{1}{\sqrt{2}}(-a^\dagger_{1,j+1} + a^\dagger_{2,j})$ we observe that it is a product state defined by an equal weight superposition (or spin x-state) defined along the solid bonds in* [Figure 4.2]. *Conversely, $|R\rangle = \prod_j r^\dagger_{2,j}|\,vac.\,\rangle$ is a product state of spin-x hybridizations along the dashed bonds. Second, the states $|L\rangle$, $|R\rangle$ afford an interpretation of free fermion ground states of distinct topological order. (...)* “

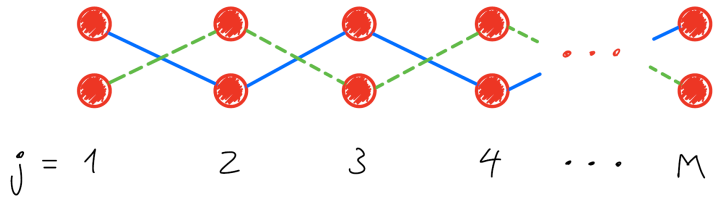


Figure 4.2: Visualization of the M-quasiparticle-states $|\,\mathrm{L}\,\rangle$ (blue) and $|\,\mathrm{R}\,\rangle$ (green).

The concept of topological order is briefly introduced in the following.

4.1.2 Topology

Corollary 4.4 The states

$$|\,\mathrm{L}\,\rangle = \prod_{j=1}^{M} l^\dagger_{2,j}|\,\mathrm{vac.}\,\rangle, \qquad |\,\mathrm{R}\,\rangle = \prod_{j=1}^{M} r^\dagger_{2,j}|\,\mathrm{vac.}\,\rangle \tag{4.2}$$

are the ground-states of the Hamiltonians[4]

$$H_\mathrm{L} = \sum_j \begin{pmatrix} l_{1,j} \\ l_{2,j} \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} l_{1,j} \\ l_{2,j} \end{pmatrix}, \qquad H_\mathrm{R} = \sum_j \begin{pmatrix} r_{1,j} \\ r_{2,j} \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} r_{1,j} \\ r_{2,j} \end{pmatrix}. \tag{4.3}$$

[4] In this case, one also speaks about so-called “parent Hamiltonians” [15], [16].

Further, using Definition 4.2, these Hamiltonians can be written as

$$H_{\mathrm{L}} = \sum_q \begin{pmatrix} a_{1,q} \\ a_{2,q} \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathrm{e}^{+\mathrm{i}\,q\,a} \\ \mathrm{e}^{-\mathrm{i}\,q\,a} & 0 \end{pmatrix} \begin{pmatrix} a_{1,q} \\ a_{2,q} \end{pmatrix} \qquad H_{\mathrm{R}} = \sum_q \begin{pmatrix} a_{1,q} \\ a_{2,q} \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\,q\,a} \\ \mathrm{e}^{+\mathrm{i}\,q\,a} & 0 \end{pmatrix} \begin{pmatrix} a_{1,q} \\ a_{2,q} \end{pmatrix}. \tag{4.4}$$

Proof 4.4 See Appendix, Proofs, page 83.

A look into a basic course on topological insulators (e. g. [26]) reveals that one considers Hamiltonian operators of this shape more often to introduce so-called "winding-numbers":

Notation 4.5 One expands the Hamiltonians as

$$H_\bullet(q) = h_\bullet^0(q)\,\sigma_0 + \vec{h}_\bullet(q) \cdot \vec{\sigma} \tag{4.5}$$

and defines the so-called winding-number $W_\bullet$ as integral

$$W_\bullet := \frac{1}{2\pi} \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \left(\frac{\vec{h}_\bullet(q)}{h_\bullet(q)} \times \frac{\mathrm{d}}{\mathrm{d}q} \frac{\vec{h}_\bullet(q)}{h_\bullet(q)} \right)_z \mathrm{d}q. \tag{4.6}$$

Theorem 4.6 In our special cases we have

$$W_{\mathrm{L}} = -1, \qquad W_{\mathrm{R}} = +1. \tag{4.7}$$

Proof 4.6 See Appendix, Proofs, page 85.

We thus obtain two ground-states of the respective parent Hamiltonians which are in some sense[5] "similar", with different winding-numbers.

Recalling now that the definition of a topological phase transition is the abrupt change of the winding-number [27], it is now reasonable to assume that these two *different* ground-states correspond *in reality* merely to two different topological phases of the very same system. Furthermore, we postulate that these two ground-states can be transformed into each other (experimentally and in particular also on paper).

The goal is now to model this phase transition by connecting the different operator sets sufficiently smooth with each other. The most obvious approach would be to connect the operators by a "straight line"[6]. However, it has proven fruitful to use sine- and cosine-functions instead. This will lead to many simplifications using trigonometric identities.

[5] Both ground-states correspond to Slater-determinants of "hybridized" wave-functions.

[6] We mean the mapping $[0,1] \ni t \mapsto (1-t)\,l + t\,x$.

4.1.3 Identities

Definition 4.7 We connect the operators l and r by introducing a parameter $\theta \in [0, \pi]$ and operators $x \equiv x(\theta)$ as

$$x := l \cdot \cos\theta + r \cdot \sin\theta,$$

where we usually omit the θ-dependence for notational simplicity.

Theorem 4.8 One immediately concludes that

$$x_{\nu,j} = \frac{\cos\theta}{\sqrt{2}} \left((-1)^{\nu+1} a_{1,j+1} + a_{2,j}\right) + \frac{\sin\theta}{\sqrt{2}} \left((-1)^{\nu+1} a_{1,j} + a_{2,j+1}\right).$$

Additionally, these newly defined operators satisfy the following commutation relations

$$\{x_{\nu,j}, x^\dagger_{\nu',j'}\} = \delta_{\nu,\nu'} \left(\delta_{j,j'} + \frac{1}{2}\sin(2\theta)(\delta_{j,j'+1} + \delta_{j+1,j'})\right), \qquad \{x_{\nu,j}, x_{\nu',j'}\} = \{x^\dagger_{\nu,j}, x^\dagger_{\nu',j'}\} = 0.$$

Proof 4.8 See Appendix, Proofs, page 85.

Theorem 4.9 For $q \in \{q_1, \ldots, q_M\}$ one obtains the Fourier-transformed operators

$$\begin{aligned} x_{\nu,q} &:= \frac{1}{\sqrt{M}} \sum_j \mathrm{e}^{\mathrm{i}qx_j} x_{\nu,j} \\ &= (-1)^{\nu+1} \frac{1}{\sqrt{2}} \left(\cos\theta\, \mathrm{e}^{-\mathrm{i}qa} + \sin\theta\right) a_{1,q} + \frac{1}{\sqrt{2}} \left(\cos\theta + \sin\theta\, \mathrm{e}^{-\mathrm{i}qa}\right) a_{2,q} \\ &\equiv (-1)^{\nu+1} c_1\, a_{1,q} + c_2\, a_{2,q}. \end{aligned} \tag{4.8}$$

Further it follows that

$$\{x_{\nu,q}, x_{\nu',q'}\} = \{x^\dagger_{\nu,q}, x^\dagger_{\nu',q'}\} = 0, \qquad \{x_{\nu,q}, x^\dagger_{\nu',q'}\} = \delta_{\nu,\nu'}\delta_{q,q'} \left(1 + \sin(2\theta)\cos(qa)\right) \equiv \delta_{\nu,\nu'}\delta_{q,q'}\, N(q,\theta)^2.$$

Proof 4.9 See Appendix, Proofs, page 86.

So we see that the CAR are satisfied except for a factor that depends on q and θ. To obtain normalized states, we define new operators that differ from the above operators by a factor as follows.

Definition 4.10 We introduce fermionic operators as

$$d_{\nu,q}(\theta) := \frac{x_{\nu,q}(\theta)}{N(q,\theta)}, \qquad \text{etc.}$$

satisfying the CAR

$$\{d_{\nu,q}, d^\dagger_{\nu',q'}\} = \delta_{\nu,\nu'}\delta_{q,q'}, \qquad \{d_{\nu,q}, d_{\nu',q'}\} = \{d^\dagger_{\nu,q}, d^\dagger_{\nu',q'}\} = 0.$$

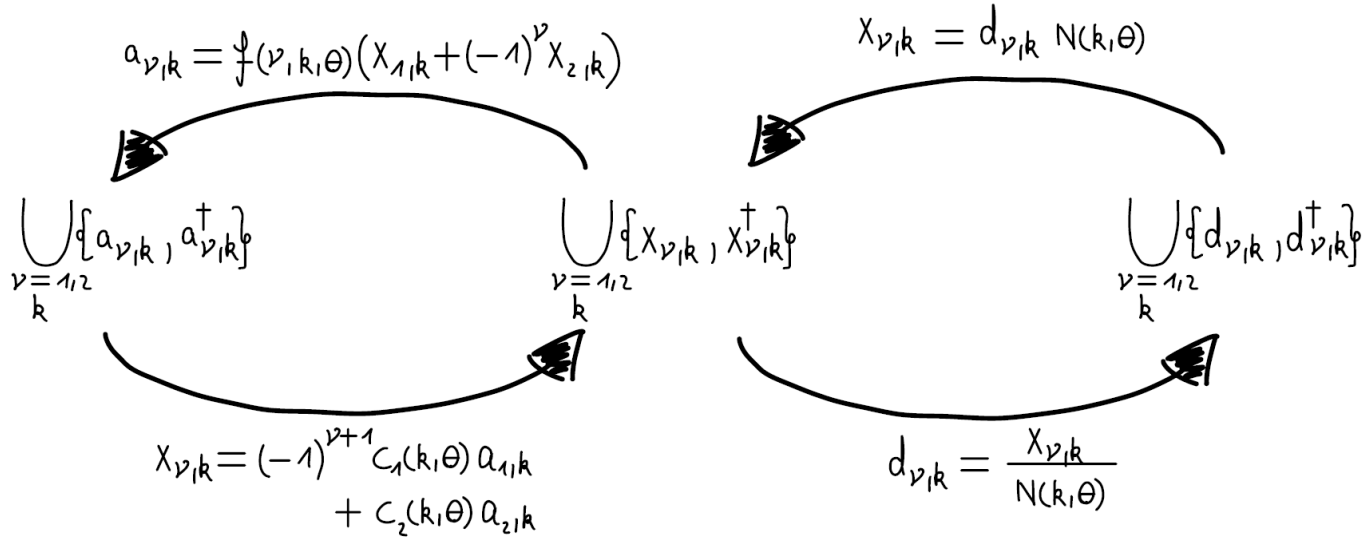


Figure 4.3: Sketch illustrating the different operators. The operators $a^\dagger$ create particles occupying Wannier-states, while the operators $d^\dagger$ create quasi-particles occupying "hybridized orbitals".

Remark 4.10.1 A few useful identities are as follows

$$N(q,\theta)^2 = 1 + \sin(2\theta)\cos(qa),$$

$$|f(1,q,\theta)|^2 = |f(2,q,\theta)|^2 \equiv |f(q,\theta)|^2 = \frac{1}{2}\frac{1}{N(q,\theta)^2},$$

$$c_1(k,\theta) = \frac{1}{\sqrt{2}}\left(\cos\theta\, \mathrm{e}^{-\mathrm{i}ka} + \sin\theta\right), \qquad c_2(k,\theta) = \frac{1}{\sqrt{2}}\left(\cos\theta + \sin\theta\, \mathrm{e}^{-\mathrm{i}ka}\right),$$

$$f(1,q,\theta) = \frac{1}{\sqrt{2}}\frac{\cos\theta\cos qa + \sin\theta + \mathrm{i}\cos\theta\sin qa}{N(q,\theta)^2}, \qquad f(2,q,\theta) = \frac{1}{\sqrt{2}}\frac{\cos\theta + \sin\theta\cos qa + \mathrm{i}\sin\theta\sin qa}{N(q,\theta)^2}.$$

$$\frac{f^*(1,q,\theta)f(2,q,\theta)}{|f(q,\theta)|^2} = \frac{\sin(2\theta)+\cos(qa)}{N(q,\theta)^2} + \mathrm{i}\sin(qa)\frac{\sin^2(\theta)-\cos^2(\theta)}{N(q,\theta)^2} \overset{\theta=\pi/4}{=} 1$$

Proof 4.10.1 See Appendix, Proofs, page 86.

Theorem 4.11 The state

$$|\mathrm{D}(\theta)\rangle \equiv |\mathrm{D}\rangle := \prod_{j=1}^{M} d_{2,j}^{\dagger}|\mathrm{vac.}\rangle \tag{4.9}$$

satisfies the equations

$$\begin{aligned}
&\text{(i)} \quad |\mathrm{D}(0)\rangle &&= \prod_{j=1}^{M} l_{2,j}^{\dagger}|\mathrm{vac.}\rangle = |\mathrm{L}\rangle,\\
&\text{(ii)} \quad |\mathrm{D}(\pi/2)\rangle &&= \prod_{j=1}^{M} r_{2,j}^{\dagger}|\mathrm{vac.}\rangle = |\mathrm{R}\rangle,\\
&\text{(iii)} \quad |\mathrm{D}\rangle &&= \prod_{k} d_{2,k}^{\dagger}|\mathrm{vac.}\rangle := d_{2,k_1}^{\dagger} d_{2,k_2}^{\dagger} \cdots d_{2,k_M}^{\dagger}|\mathrm{vac.}\rangle.
\end{aligned}$$

Proof 4.11 See Appendix, Proofs, page 87.

For later reference, we note the trivial expectation values:

Corollary 4.12 The following equations for real-space and momentum-space occupation number expectation values are immediate from the above theorem

$$\begin{aligned}
&\langle \mathrm{D} \,|\, d_{1,j}^{\dagger} d_{1,j'} \,|\, \mathrm{D} \rangle = 0, && \langle \mathrm{D} \,|\, d_{2,j}^{\dagger} d_{2,j'} \,|\, \mathrm{D} \rangle = \delta_{j,j'},\\
&\langle \mathrm{D} \,|\, d_{1,j} d_{1,j'}^{\dagger} \,|\, \mathrm{D} \rangle = \delta_{j,j'}, && \langle \mathrm{D} \,|\, d_{2,j} d_{2,j'}^{\dagger} \,|\, \mathrm{D} \rangle = 0,\\
&\langle \mathrm{D} \,|\, d_{1,k}^{\dagger} d_{1,k'} \,|\, \mathrm{D} \rangle = 0, && \langle \mathrm{D} \,|\, d_{2,k}^{\dagger} d_{2,k'} \,|\, \mathrm{D} \rangle = \delta_{k,k'},\\
&\langle \mathrm{D} \,|\, d_{1,k} d_{1,k'}^{\dagger} \,|\, \mathrm{D} \rangle = \delta_{k,k'}, && \langle \mathrm{D} \,|\, d_{2,k} d_{2,k'}^{\dagger} \,|\, \mathrm{D} \rangle = 0.
\end{aligned}$$

4.2 Introducing a concrete Lindblad-Equation

Definition 4.13 For $\alpha = (j, \nu, \mu) \in \{1, \ldots, M\} \times \{1, 2\} \times \{1, 2\}$ we define[7] the operators

$$L_{\alpha} = \delta_{\mu,1}(\delta_{\nu,1} a_{1,j}^{\dagger} x_{1,j} + \delta_{\nu,2} a_{2,j}^{\dagger} x_{1,j}) + \delta_{\mu,2}(\delta_{\nu,1} a_{1,j} x_{2,j}^{\dagger} + \delta_{\nu,2} a_{2,j} x_{2,j}^{\dagger}) \tag{4.10}$$

and introduce the Lindblad-equation

$$\partial_t \rho = \sum_{\alpha} \kappa \left(2 L_{\alpha} \rho L_{\alpha}^{\dagger} - \{ L_{\alpha}^{\dagger} L_{\alpha}, \rho \} \right) \equiv \mathscr{L}[\rho].$$

[7] In the template [15], the quantum jump operators were introduced in a different way. The proof that both representations are equivalent is given in Appendix, Proofs, page 88.

Remark 4.13.1 These operators satisfy a whole set of physically meaningful properties:

- Since $d \propto x$, one recognizes immediately that $L_\alpha|\mathrm{D}\rangle = 0$ holds. This means that the state $\rho = |\mathrm{D}\rangle\langle\mathrm{D}|$ is a stationary state that solves the above Lindblad-equation. Thus, the state $|\mathrm{D}\rangle$ is a *dark state* of the Lindblad-equation.
- Because L_α is a monomial consisting of exactly one creation- and one annihilation-operator, it follows (in the language of second quantization) that the operators L_α are *particle number conserving*.
- In particular, the operators L_α act "locally": If a quasi-particle is annihilated or created at some position $r_j \equiv j \cdot a$, then another superposition of quasi-particles is also created or annihilated at this or the nearest neighbors position. In this spirit, the quasi-particles "jump" under the influence of the interaction between lattice sites, illuminating the name "quantum *jump* operators".

Remark 4.13.2 The quantum jump operators were introduced in formal correspondence to a topological insulator; the dark state which solves the above Lindblad-equation is at the same time a ground-state of a topological insulator. This is a common way to introduce Lindblad-equations. (See, for example, [17], [9].)

5 Approximations and Functional Integral Formalism

Let's recall from section 3.1.2 that we have at least two possibilities to determine the partition function Z corresponding to the formal solution of the Lindblad-equation:

We can ...

1. use the usual definition as a trace over a Fock-space-basis $\{|I\rangle\}$ as

$$Z \equiv Z_{\mathrm{Op}} = \mathrm{Tr}\, \rho(t) = \sum_I \langle I|\rho(t)|I\rangle.$$

2. use Theorem 3.3 and Hypothesis 3.7 to consider a functional field integral (in the literature [28] also known as [K]eldysh functional integral)

$$Z \equiv Z_{\mathrm{K}} \approx \int \mathscr{D}(\psi, \overline{\psi}) \exp\, - S[\psi, \overline{\psi}].$$

Our ultimate goal is of course to study the phase transition of the system from Definition 4.13 with quantum field theoretical methods from chapter 3. As the action of the system of interest has non-trivial quartic contributions[1] we need to resort to approximations.

5.1 Self-Energy in First Order Perturbation Theory

In this section we will use seemingly different, but mutually equivalent approximations from [16] and [17] (mean field theory and Born-Approximation) to prove the commutativity of the diagram

$$\begin{array}{ccc} Z_{\mathrm{Op}} & \xrightarrow{\text{Mean Field Theory}} & Z_{\mathrm{Op}}^{\mathrm{MF}} \\ \Big\downarrow \text{QFT} & & \Big\downarrow \text{QFT} \\ Z_{\mathrm{K}} & \xrightarrow{\text{Born-Approximation}} & Z_{\mathrm{K}}^{\mathrm{BA}} = Z_{\mathrm{K}}^{\mathrm{MF}} \end{array}$$

where the individual approximations are introduced in the next sections.

[1] This can be seen from the definition of the quantum jump operators in eq. (4.10) and the formula for the action in eq. (3.3)

5.1.1 Mean Field Theory of the Modes in the Operator Formalism

Approximation in the Operator Formalism

We use the same strategy as in the appendix of [16] to find an effective Liouville-operator for a single mode $p \in \{q_1, \ldots, q_M\}$.

Approximation 5.1 In the mean field approximation we make the following ansatz

$$\rho \overset{\text{Ansatz}}{\approx} \bigotimes_q \rho_q := \rho_{q_1} \otimes \cdots \otimes \rho_{q_M},$$

where the density ρ_q describes the mode q, and compute the partial traces

$$\mathrm{Tr}_{\neq p}\, \mathscr{L}[\rho].$$

The resulting expectation values are evaluated with respect to the dark state $|\mathrm{D}\rangle$ (which solves the stationary Lindblad-equation), and the terms which are quartic in creation- and annihilation-operators of the mode p are neglected.

Theorem 5.2 The mean field approximation yields

$$\mathrm{Tr}_{\neq p}\, \mathscr{L}[\rho] \quad \approx \quad 2\,\mathrm{K}\,(d_{1,p}\,\rho_p\,d_{1,p}^\dagger + d_{2,p}^\dagger\,\rho_p\,d_{2,p}) - \mathrm{K}\,\{d_{1,p}^\dagger d_{1,p} + d_{2,p} d_{2,p}^\dagger\,,\,\rho_p\},$$

with

$$\mathrm{K} \equiv \mathrm{K}(p,\theta) = \kappa \cdot N^2(p,\theta) = \kappa(1 + \sin(2\theta)\cos(pa)). \tag{5.1}$$

Before proving this theorem, let's elucidate what it implies.

Corollary 5.3 In the mean field approximation, we obtain an approximated Lindblad-equation for the density ρ:

$$\partial_t \rho \approx \sum_q \sum_{j=1}^{2} \mathrm{K}(q,\theta)\left(2L_{q,j}\rho L_{q,j}^\dagger - \{L_{q,j}^\dagger L_{q,j}\,,\,\rho\}\right) =: \mathscr{L}_{\mathrm{MF}}[\rho]$$

with

$$L_{q,j} = \begin{cases} d_{q,1} & \text{for } j = 1 \\ d_{q,2}^\dagger & \text{for } j = 2 \end{cases}.$$

Proof 5.3 Follows immediately with the product rule and Theorem 5.2.

In order to prove Theorem 5.2 we divide ("and conquer") the problem by using the linearity of the trace:

$$\mathrm{Tr}_{\neq p}\,\mathscr{L}[\rho] = \underbrace{2\,\mathrm{Tr}_{\neq p}\left[\sum_\alpha L_\alpha \rho L_\alpha^\dagger\right]}_{=:\,\mathrm{I}} - \underbrace{\mathrm{Tr}_{\neq p}\left[\sum_\alpha L_\alpha^\dagger L_\alpha \rho\right]}_{=:\,\mathrm{II}} - \underbrace{\mathrm{Tr}_{\neq p}\left[\sum_\alpha \rho L_\alpha^\dagger L_\alpha\right]}_{=:\,\mathrm{III}}$$

Lemma 5.4 For contribution I one gets

$$\begin{aligned}
&2\,\mathrm{Tr}_{\neq p}\left[\sum_\alpha L_\alpha \rho L_\alpha^\dagger\right] \\
&= \frac{2\kappa}{M}\sum_{\nu\in\{1,2\}}\mathrm{Tr}_{\neq p}\sum_{q_1,q_2,q_3,q_4}\delta_{q_1+q_3-(q_2+q_4)}f^*(\nu,q_1,\theta)f(\nu,q_4,\theta)\{x^\dagger_{1,q_1}x_{1,q_2}\,\rho\,x^\dagger_{1,q_3}x_{1,q_4} \\
&\qquad + (-1)^\nu x^\dagger_{1,q_1}x_{1,q_2}\,\rho\,x^\dagger_{1,q_3}x_{2,q_4} \\
&\qquad + (-1)^\nu x^\dagger_{2,q_1}x_{1,q_2}\,\rho\,x^\dagger_{1,q_3}x_{1,q_4} \\
&\qquad + x^\dagger_{2,q_1}x_{1,q_2}\,\rho\,x^\dagger_{1,q_3}x_{2,q_4}\} \\
&+ \frac{2\kappa}{M}\sum_{\nu\in\{1,2\}}\mathrm{Tr}_{\neq p}\sum_{q_1,q_2,q_3,q_4}\delta_{q_1+q_3-(q_2+q_4)}f(\nu,q_1,\theta)f^*(\nu,q_4,\theta)\{x_{1,q_1}x^\dagger_{2,q_2}\,\rho\,x_{2,q_3}x^\dagger_{1,q_4} \\
&\qquad + (-1)^\nu x_{1,q_1}x^\dagger_{2,q_2}\,\rho\,x_{2,q_3}x^\dagger_{2,q_4} \\
&\qquad + (-1)^\nu x_{2,q_1}x^\dagger_{2,q_2}\,\rho\,x_{2,q_3}x^\dagger_{1,q_4} \\
&\qquad + x_{2,q_1}x^\dagger_{2,q_2}\,\rho\,x_{2,q_3}x^\dagger_{2,q_4}\} \\
&=: \sum_{\nu\in\{1,2\}}\mathrm{I}_1(\nu)+\mathrm{I}_2(\nu)+\ldots+\mathrm{I}_8(\nu),
\end{aligned}$$

where in the last equality the numbering is from top to bottom.

Proof 5.4 See Appendix, Proofs, page 89.

Lemma 5.5 Evaluating the individual expectation values according to the approximation and neglecting terms that are quartic in the creation- and annihilation-operators (of the mode p) gives

$$\mathrm{I}_j(\nu) \approx \begin{cases} \mathrm{K}\; d_{1,p}\,\rho_p\,d^\dagger_{1,p} & \text{for } j=1 \text{ and } \nu\in\{1,2\} \\ \mathrm{K}\; d^\dagger_{2,p}\,\rho_p\,d_{2,p} & \text{for } j=8 \text{ and } \nu\in\{1,2\} \\ 0 & \text{else.} \end{cases}.$$

Thus one receives for contribution I the equality

$$\mathrm{I} = 2\,\mathrm{Tr}_{\neq p}\left[\sum_\alpha L_\alpha \rho L_\alpha^\dagger\right] \approx 2\,\mathrm{K}\left(d_{1,p}\,\rho_p\,d^\dagger_{1,p} + d^\dagger_{2,p}\,\rho_p\,d_{2,p}\right).$$

Proof 5.5 See Appendix, Proofs, page 89.

Lemma 5.6 For contribution II one gets

$$
\begin{aligned}
&\mathrm{Tr}_{\neq p}\left[\sum_{\alpha} L_{\alpha}^{\dagger} L_{\alpha}\rho\right] \\
&= \frac{\kappa}{M}\sum_{\nu\in\{1,2\}}\mathrm{Tr}_{\neq p}\sum_{q_1,q_2,q_3,q_4}\delta_{q_1+q_3-(q_2+q_4)} f(\nu,q_2,\theta) f^*(\nu,q_3,\theta)\{x_{1,q_1}^{\dagger}x_{1,q_2}x_{1,q_3}^{\dagger}x_{1,q_4}\,\rho \\
&\qquad\qquad + (-1)^{\nu}x_{1,q_1}^{\dagger}x_{2,q_2}x_{1,q_3}^{\dagger}x_{1,q_4}\,\rho \\
&\qquad\qquad + (-1)^{\nu}x_{1,q_1}^{\dagger}x_{1,q_2}x_{2,q_3}^{\dagger}x_{1,q_4}\,\rho \\
&\qquad\qquad + x_{1,q_1}^{\dagger}x_{2,q_2}x_{2,q_3}^{\dagger}x_{1,q_4}\,\rho\} \\
&+ \frac{\kappa}{M}\sum_{\nu\in\{1,2\}}\mathrm{Tr}_{\neq p}\sum_{q_1,q_2,q_3,q_4}\delta_{q_1+q_3-(q_2+q_4)} f^*(\nu,q_2,\theta) f(\nu,q_3,\theta)\{x_{2,q_1}x_{1,q_2}^{\dagger}x_{1,q_3}x_{2,q_4}^{\dagger}\,\rho \\
&\qquad\qquad + (-1)^{\nu}x_{2,q_1}x_{2,q_2}^{\dagger}x_{1,q_3}x_{2,q_4}^{\dagger}\,\rho \\
&\qquad\qquad + (-1)^{\nu}x_{2,q_1}x_{1,q_2}^{\dagger}x_{2,q_3}x_{2,q_4}^{\dagger}\,\rho \\
&\qquad\qquad + x_{2,q_1}x_{2,q_2}^{\dagger}x_{2,q_3}x_{2,q_4}^{\dagger}\,\rho\} \\
&=: \sum_{\nu\in\{1,2\}}\mathrm{II}_1(\nu)+\mathrm{II}_2(\nu)+\ldots+\mathrm{II}_8(\nu),
\end{aligned}
$$

where in the last equality the numbering is from top to bottom.

Proof 5.6 Is done by renumbering and reordering the individual terms in Lemma 5.4.

Lemma 5.7 Evaluating the individual expectation values according to the approximation and neglecting terms that are quartic in the creation- and annihilation-operators (of the mode p) gives

$$
\mathrm{II}_j(\nu) \approx \begin{cases} \mathrm{K}\,\frac{1}{2}\,d_{1,p}^{\dagger}d_{1,p}\,\rho_p & \text{for } j=1 \text{ and } \nu\in\{1,2\} \\ \mathrm{K}\,\frac{1}{2}\,d_{2,p}d_{2,p}^{\dagger}\,\rho_p & \text{for } j=8 \text{ and } \nu\in\{1,2\} \\ 0 & \text{else.} \end{cases}.
$$

Thus one receives for contribution II the equality

$$
\mathrm{II} = \mathrm{Tr}_{\neq p}\left[\sum_{\alpha} L_{\alpha}^{\dagger}L_{\alpha}\rho\right] \approx \mathrm{K}\left(d_{1,p}^{\dagger}d_{1,p}\,\rho_p + d_{2,p}d_{2,p}^{\dagger}\,\rho_p\right).
$$

Proof 5.7 See Appendix, Proofs, page 92.

If we now combine the results of Lemma 5.5 and Lemma 5.7 and use the cyclicity of the trace (to determine III), then one obtains the result announced above.

Transition to the Functional Integral in the Mean Field Theory of the Modes

In section 3.1.2 it was shown that the partition function corresponding to the formal solution of a Lindblad-equation can be written in the functional integral formalism as

$$Z \approx \int \mathscr{D}(\psi, \overline{\psi}) \exp - S[\psi, \overline{\psi}],$$

where

$$-S[\psi, \overline{\psi}] \equiv \int \mathrm{d}t \left[(\partial_t \overline{\psi}_+) \cdot \psi_+ + \overline{\psi}_- \cdot \partial_t \psi_- + \sum_\alpha 2\, L_{\alpha,+} L^\dagger_{\alpha,-} - \left((L^\dagger_\alpha L_\alpha)_+ + (L^\dagger_\alpha L_\alpha)_- \right) \right].$$

This formula can now be applied (via the various variable transformations of the Graßmann-numbers) to our concrete exact system. Here we present the application to the approximated system.

Corollary 5.8 For the approximated Lindblad-equation

$$\partial_t \rho \approx \sum_q \sum_{j=1}^{2} \mathrm{K}(q, \theta) \left(2 L_{q,j} \rho L^\dagger_{q,j} - \{ L^\dagger_{q,j} L_{q,j} \,,\, \rho \} \right) =: \mathscr{L}_{\mathrm{MF}}[\rho],$$

with

$$L_{q,j} = \begin{cases} d_{q,j} & \text{for } j = 1 \\ d^\dagger_{q,j} & \text{for } j = 2 \end{cases},$$

we obtain (in the functional integral formalism together with the variable transformations in Figure 5.1) the action

$$\begin{aligned} -S_{\mathrm{MF}}[\xi, \overline{\xi}] \equiv \int \mathrm{d}t \sum_q \Big[& (\partial_t \overline{\xi}_{+,1})\, \xi_{+,1} + \overline{\xi}_{-,1}\, \partial_t\, \xi_{-,1} + \mathrm{K}(\theta) \big(2\, \overline{\xi}_{-,1} \xi_{+,1} - \overline{\xi}_{+,1} \xi_{+,1} - \overline{\xi}_{-,1} \xi_{-,1} \big) \\ & + (\partial_t \overline{\xi}_{+,2})\, \xi_{+,2} + \overline{\xi}_{-,2}\, \partial_t\, \xi_{-,2} + \mathrm{K}(\theta) \big(-2\, \overline{\xi}_{+,2} \xi_{-,2} + \overline{\xi}_{+,2} \xi_{+,2} + \overline{\xi}_{-,2} \xi_{-,2} \big) \Big] \\ \equiv \quad & \oint_{t,q} \begin{pmatrix} \overline{\xi}_{+,1} \\ \overline{\xi}_{-,1} \\ \overline{\xi}_{+,2} \\ \overline{\xi}_{-,2} \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} \overleftarrow{\partial}_t - \mathrm{K}(\theta) & 0 & & \\ 2\,\mathrm{K}(\theta) & \partial_t - \mathrm{K}(\theta) & & \\ & & \overleftarrow{\partial}_t + \mathrm{K}(\theta) & -2\,\mathrm{K}(\theta) \\ & & 0 & \partial_t + \mathrm{K}(\theta) \end{pmatrix} \cdot \begin{pmatrix} \xi_{+,1} \\ \xi_{-,1} \\ \xi_{+,2} \\ \xi_{-,2} \end{pmatrix}, \end{aligned}$$

with $\kappa\, N^2 \equiv \kappa\, N^2(q, \theta) =: \mathrm{K}(q, \theta) \equiv \mathrm{K}(\theta)$ and $\xi_{+,b} \equiv \xi_{+,b}(t, q)$ where $b \in \{1, 2\}$ corresponds to the energy band.

Proof 5.8 Follows immediately by using the definitions.

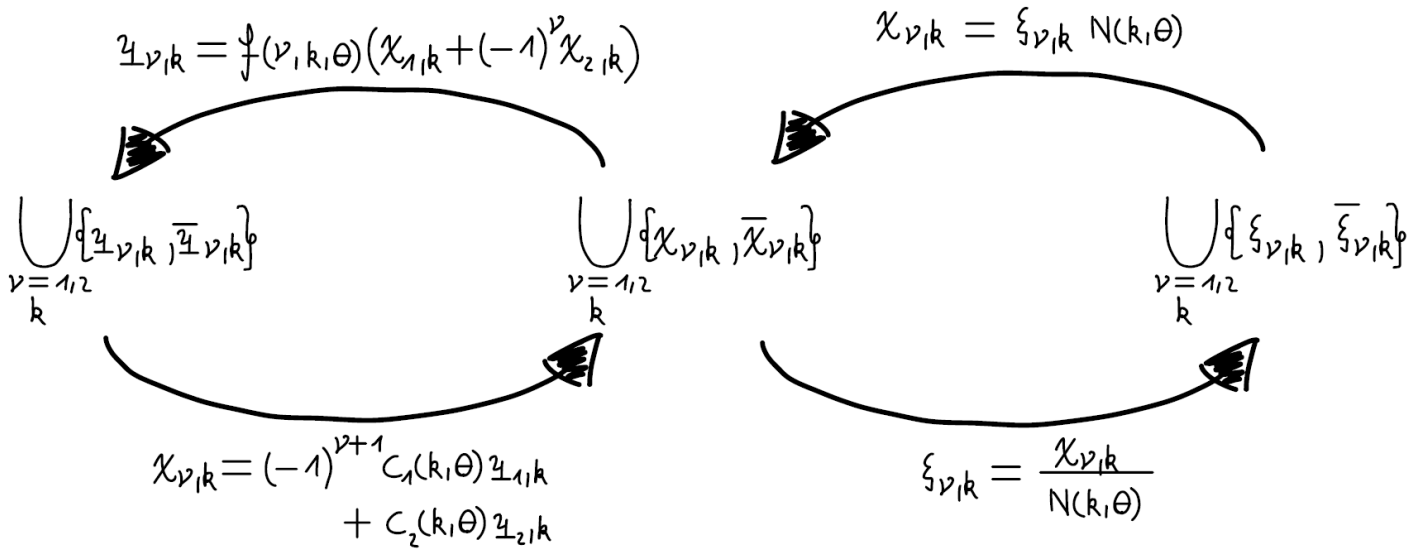


Figure 5.1: Sketch illustrating the different Graßmann-numbers; These variable transformations correspond exactly to the base changes in Figure 4.3.

5.1.2 Born-Approximation in the Functional Integral Formalism

The goal is now to reproduce the result of the approximation in the operator formalism within the framework of another approximation in the functional integral formalism (indicating the validity of the approximations).

We again consider the exact Lindblad-equation from Definition 4.13

$$\partial_t \rho = \sum_\alpha \kappa \left(2L_\alpha \rho L_\alpha^\dagger - \{L_\alpha^\dagger L_\alpha, \rho\}\right) \equiv \mathscr{L}[\rho]$$

with quantum jump operators

$$L_\alpha = \delta_{\mu,1}(\delta_{\nu,1} a_{1,j}^\dagger x_{1,j} + \delta_{\nu,2} a_{2,j}^\dagger x_{1,j}) + \delta_{\mu,2}(\delta_{\nu,1} a_{1,j} x_{2,j}^\dagger + \delta_{\nu,2} a_{2,j} x_{2,j}^\dagger).$$

where $\alpha = (j, \nu, \mu) \in \{1, \dots, M\} \times \{1,2\} \times \{1,2\}$.

Approximation 5.9 The action in eq. (3.3) for the exact Lindblad-equation can be simplified using the Born-approximation from [17] reproducing the action in Corollary 5.8.

Proof 5.9 The basic idea is to approximate the quartic terms in the action by weighted bilinears, and to smartly avoid the normal-ordering-procedure by introducing additional timestamps. The philosophy of this approximation and further details are given in Appendix, Proofs, page 94.

5.2 Approximated Correlators and Assessment of the Approximations

We follow the strategies in [18] and realize that the approximated correlators are analytically accessible, where we restrict ourselves (for notational simplicity) to the case $t_\mathrm{i} = 0$ and $t_\mathrm{f} = T$.

5.2.1 Momentum-, Time-Correlators

Theorem 5.10 One has for the first band

- $G_1^{<}(q,t_i,k,t_j) \equiv \langle \xi_{+,1}(q,t_i)\overline{\xi}_{-,1}(k,t_j)\rangle = -\dfrac{\delta_{q,k}}{2\exp\left(|t_i+t_j|\,\mathrm{K}(q,\theta)\right)}$,
- $G_1^{>}(q,t_i,k,t_j) \equiv \langle \xi_{-,1}(q,t_i)\overline{\xi}_{+,1}(k,t_j)\rangle = \dfrac{\delta_{q,k}}{\exp\left(|t_i-t_j|\,\mathrm{K}(q,\theta)\right)} - \dfrac{\delta_{q,k}}{2\exp\left(|t_i+t_j|\,\mathrm{K}(q,\theta)\right)}$,
- $G_1^{\mathbb{T}}(q,t_i,k,t_j) \equiv \langle \xi_{+,1}(q,t_i)\overline{\xi}_{+,1}(k,t_j)\rangle = \dfrac{\delta_{q,k}\,\theta(t_i-t_j)}{\exp\left(|t_i-t_j|\,\mathrm{K}(q,\theta)\right)} - \dfrac{\delta_{q,k}}{2\exp\left(|t_i+t_j|\,\mathrm{K}(q,\theta)\right)}$,
- $G_1^{\widetilde{\mathbb{T}}}(q,t_i,k,t_j) \equiv \langle \xi_{-,1}(q,t_i)\overline{\xi}_{-,1}(k,t_j)\rangle = \dfrac{\delta_{q,k}\,\theta(t_j-t_i)}{\exp\left(|t_i-t_j|\,\mathrm{K}(q,\theta)\right)} - \dfrac{\delta_{q,k}}{2\exp\left(|t_i+t_j|\,\mathrm{K}(q,\theta)\right)}$,

and similar identities for the second band.

Proof 5.10 See Appendix, Proofs, page 97.

Assessment of the Approximations

First, we note that the presence of $\mathrm{K}(q,\theta)$ in the propagators is completely intuitive: the temporal evolution of the system is affected by the effective rate of the interaction.

Second, we mention that our mean field action passes the "sanity-check" from Corollary 3.6, since one has for each band

$$\overleftarrow{\partial}_t - \mathrm{K}(q,\theta) \quad + \quad 2\,\mathrm{K}(q,\theta) \quad + \quad \partial_t - \mathrm{K}(q,\theta) \equiv 0.$$

Third, the approximations are self-consistent in the sense that we reach the dark state $|\,\mathrm{D}\,\rangle$ from eq. (4.9) in the limit of "long times". We put the following facts together to form an overall picture:

(1) The momentum-, time-correlators for the approximated quadratic action were analytically calculated in Theorem 5.10.

(2) Corollary 3.8 links expectation values with respect to the formal solution in eq. (3.2) to correlation functions for "long times".

(3) In Hypothesis 3.2 we mentioned that we are optimistic about the identity between the formal solution in eq. (3.2) and the exact solution in eq. (4.9) in the case of "long times".

(4) In Corollary 4.12 we gave some expectation values for the occupation numbers of the dark state.

Our approximations are consistent with all these considerations, because for $|N-\widetilde{N}| \leq 1$ we obtain

$$\begin{aligned}
0 &\overset{(1)}{\approx} G_{1,q}^{<}(t_N,t_N) \overset{(2)}{=} \langle d_{1,q}^{\dagger} d_{1,q}\rangle && \overset{(3)}{=} \langle\,\mathrm{D}\,|\,d_{1,q}^{\dagger} d_{1,q}\,|\,\mathrm{D}\,\rangle && \overset{(4)}{=} 0,\\
1 &\overset{(1)}{\approx} G_{1,q}^{>}(t_N,t_N) \overset{(2)}{=} 1-\langle d_{1,q}^{\dagger} d_{1,q}\rangle && \overset{(3)}{=} 1-\langle\,\mathrm{D}\,|\,d_{1,q}^{\dagger} d_{1,q}\,|\,\mathrm{D}\,\rangle && \overset{(4)}{=} 1,\\
\delta_{N,\widetilde{N}} &\overset{(1)}{\approx} G_{1,q}^{\mathbb{T}}(t_N,t_{\widetilde{N}}) \overset{(2)}{=} \mathbb{1}_{N=\widetilde{N}}\cdot\left(1-\langle d_{1,q}^{\dagger} d_{1,q}\rangle\right) - \mathbb{1}_{N<\widetilde{N}}\cdot\langle d_{1,q}^{\dagger} d_{1,q}\rangle && \overset{(3)}{=} \mathbb{1}_{N=\widetilde{N}}\cdot\left(1-\langle\,\mathrm{D}\,|\,d_{1,q}^{\dagger} d_{1,q}\,|\,\mathrm{D}\,\rangle\right) - \mathbb{1}_{N<\widetilde{N}}\cdot\langle\,\mathrm{D}\,|\,d_{1,q}^{\dagger} d_{1,q}\,|\,\mathrm{D}\,\rangle && \overset{(4)}{=} \delta_{N,\widetilde{N}},\\
\delta_{N,\widetilde{N}} &\overset{(1)}{\approx} G_{1,q}^{\widetilde{\mathbb{T}}}(t_N,t_{\widetilde{N}}) \overset{(2)}{=} \mathbb{1}_{N=\widetilde{N}}\cdot\left(1-\langle d_{1,q}^{\dagger} d_{1,q}\rangle\right) - \mathbb{1}_{N>\widetilde{N}}\cdot\langle d_{1,q}^{\dagger} d_{1,q}\rangle && \overset{(3)}{=} \mathbb{1}_{N=\widetilde{N}}\cdot\left(1-\langle\,\mathrm{D}\,|\,d_{1,q}^{\dagger} d_{1,q}\,|\,\mathrm{D}\,\rangle\right) - \mathbb{1}_{N>\widetilde{N}}\cdot\langle\,\mathrm{D}\,|\,d_{1,q}^{\dagger} d_{1,q}\,|\,\mathrm{D}\,\rangle && \overset{(4)}{=} \delta_{N,\widetilde{N}}.
\end{aligned}$$

The terms on the leftmost side obviously agree with those on the rightmost side. Analogously all identities for the second band are fulfilled.

5.2.2 Momentum-, Frequency-Correlators

In literature as [1], [5] and [18], the following question was answered (for functional integrals similar to those discussed here) in the affirmed:

> Can one (at least in some situations), instead of dealing with the tedious details of discrete derivatives (and sums), use instead the powerful tools of differentiation (and integration) suggested by the continuum notation of the action?

Indeed, there are very many examples in which this continuum-*approximation* yields predictions that are consistent with experimental experience.[2]

Approximation 5.11 In this thesis, we are optimistic that one can approximate the time-derivatives (via a Fourier-transform neglecting infinitesimal phases) by frequency-multiplications. With this approximation the action simplifies:

$$
\begin{aligned}
-S_{\mathrm{MF}}[\xi,\bar{\xi}] \equiv \quad & \oint\!\!\!\!\!\!\sum_{t,q} \begin{pmatrix} \bar{\xi}_{+,1} \\ \bar{\xi}_{-,1} \\ \bar{\xi}_{+,2} \\ \bar{\xi}_{-,2} \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} \overleftarrow{\partial}_t - \mathrm{K}(q,\theta) & 0 & & \\ 2\,\mathrm{K}(q,\theta) & \partial_t - \mathrm{K}(q,\theta) & & \\ & & \overleftarrow{\partial}_t + \mathrm{K}(q,\theta) & -2\,\mathrm{K}(q,\theta) \\ & & 0 & \partial_t + \mathrm{K}(q,\theta) \end{pmatrix} \cdot \begin{pmatrix} \xi_{+,1} \\ \xi_{-,1} \\ \xi_{+,2} \\ \xi_{-,2} \end{pmatrix} \\
\approx \quad & \int \frac{\mathrm{d}\omega}{2\pi} \sum_{q} \begin{pmatrix} \bar{\xi}_{+,1} \\ \bar{\xi}_{-,1} \\ \bar{\xi}_{+,2} \\ \bar{\xi}_{-,2} \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} -\mathrm{i}\omega - \mathrm{K}(q,\theta) & 0 & & \\ 2\,\mathrm{K}(q,\theta) & \mathrm{i}\omega - \mathrm{K}(q,\theta) & & \\ & & -\mathrm{i}\omega + \mathrm{K}(q,\theta) & -2\,\mathrm{K}(q,\theta) \\ & & 0 & \mathrm{i}\omega + \mathrm{K}(q,\theta) \end{pmatrix} \cdot \begin{pmatrix} \xi_{+,1} \\ \xi_{-,1} \\ \xi_{+,2} \\ \xi_{-,2} \end{pmatrix} \\
\equiv \quad & \oint\!\!\!\!\!\!\sum_{\mathrm{Q}} \begin{pmatrix} \bar{\xi}_{+,1} \\ \bar{\xi}_{-,1} \\ \bar{\xi}_{+,2} \\ \bar{\xi}_{-,2} \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} -g(\theta) & 0 & & \\ (g+g^*)(\theta) & -g^*(\theta) & & \\ & & g^*(\theta) & -(g+g^*)(\theta) \\ & & 0 & g(\theta) \end{pmatrix} \cdot \begin{pmatrix} \xi_{+,1} \\ \xi_{-,1} \\ \xi_{+,2} \\ \xi_{-,2} \end{pmatrix} \\
\Longleftrightarrow +S_{\mathrm{MF}}[\xi,\bar{\xi}] \equiv \quad & \oint\!\!\!\!\!\!\sum_{\mathrm{Q}} \begin{pmatrix} \bar{\xi}_{+,1} \\ \bar{\xi}_{-,1} \\ \bar{\xi}_{+,2} \\ \bar{\xi}_{-,2} \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} g(\theta) & 0 & & \\ -(g+g^*)(\theta) & g^*(\theta) & & \\ & & -g^*(\theta) & (g+g^*)(\theta) \\ & & 0 & -g(\theta) \end{pmatrix} \cdot \begin{pmatrix} \xi_{+,1} \\ \xi_{-,1} \\ \xi_{+,2} \\ \xi_{-,2} \end{pmatrix}.
\end{aligned}
$$

where in the approximation-step the integration measure was implicitly adapted to the almost unitary transformation[3], $g(\theta) \equiv g(\theta, \mathrm{Q}) \equiv \mathrm{i}\omega + \mathrm{K}(q,\theta)$ etc.

[2] For real-valued integration-variables, one reproduces, for example, the Euler-Lagrange-equations for point-like particles from classical mechanics in the mean field theory (taking $\hbar \to 0$) of Feynmans path integral-formulation for a Schrödinger-particles propagator in one dimension. (See, for example, [1].)

For complex-valued integration-variables one reproduces, for example, the excitation spectrum of a weakly, locally and repulsively interacting Bose-gas in the Bogoliubov-approximation by a saddle point approximation. (See, for example, [5].)

For Graßmann-valued integration-variables one derives, for example, the "gap-equation" of BCS-theory after applying a Hubbard-Stratanovich-transformation, a mean field approximation, and a series expansion of the resulting action. (See, for example, [1].)

There are many more examples for each case.

[3] Prefactor-convention for the frequencies (surpressing branch- and band-indices): $\xi(t,q) = \frac{1}{T}\sum_{\omega} \mathrm{e}^{\mathrm{i}\omega t}\xi(\omega,q) \equiv \int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{\mathrm{i}\omega t}\xi(\omega,q)$. This transformation is unitary up to a (divergent) factor.

Remark 5.11.1 In the context of the above approximation it is also obvious that "integration by parts" can be carried out within the action: If one lets a time-derivative ∂_t act "to the right" on $\xi(\bullet)$, then one obtains (after changing the basis) a multiplication with $\mathrm{i}\omega$. If, on the other hand, one lets a (signed time-)derivative $(-1)\cdot\overleftarrow{\partial}_t$ act "to the left" on $\overline{\xi}(\bullet)$, one also gets (after changing the basis) a multiplication by $\mathrm{i}\omega$.

Remark 5.11.2 We will keep the discreteness of the momenta for the moment, as it's easier to detect the divergence in Lemma 5.18 this way.

Corollary 5.12 For the first band we obtain (in the "frequency space") the approximated correlators

- $G_1^<(\mathrm{Q},\mathrm{Q}') \equiv \langle \xi_{+,1}(\mathrm{Q})\overline{\xi}_{-,1}(\mathrm{Q}')\rangle = 0,$
- $G_1^>(\mathrm{Q},\mathrm{Q}') \equiv \langle \xi_{-,1}(\mathrm{Q})\overline{\xi}_{+,1}(\mathrm{Q}')\rangle = \delta_{\mathrm{Q},\mathrm{Q}'}\dfrac{(g+g^*)(\theta)}{|g(\theta)|^2} = \dfrac{\delta_{\mathrm{Q},\mathrm{Q}'}}{g(\theta)} + \dfrac{\delta_{\mathrm{Q},\mathrm{Q}'}}{g^*(\theta)},$
- $G_1^{\mathbb{T}}(\mathrm{Q},\mathrm{Q}') \equiv \langle \xi_{+,1}(\mathrm{Q})\overline{\xi}_{+,1}(\mathrm{Q}')\rangle = \dfrac{\delta_{\mathrm{Q},\mathrm{Q}'}}{g(\theta)},$
- $G_1^{\widetilde{\mathbb{T}}}(\mathrm{Q},\mathrm{Q}') \equiv \langle \xi_{-,1}(\mathrm{Q})\overline{\xi}_{-,1}(\mathrm{Q}')\rangle = \dfrac{\delta_{\mathrm{Q},\mathrm{Q}'}}{g^*(\theta)}.$

For the second band one obtains the approximated correlators

- $G_2^<(\mathrm{Q},\mathrm{Q}') = \langle \xi_{+,2}(\mathrm{Q})\overline{\xi}_{-,2}(\mathrm{Q}')\rangle = -\delta_{\mathrm{Q},\mathrm{Q}'}\dfrac{(g+g^*)(\theta)}{|g(\theta)|^2} = -\dfrac{\delta_{\mathrm{Q},\mathrm{Q}'}}{g(\theta)} - \dfrac{\delta_{\mathrm{Q},\mathrm{Q}'}}{g^*(\theta)},$
- $G_2^>(\mathrm{Q},\mathrm{Q}') = \langle \xi_{-,2}(\mathrm{Q})\overline{\xi}_{+,2}(\mathrm{Q}')\rangle = 0,$
- $G_2^{\mathbb{T}}(\mathrm{Q},\mathrm{Q}') = \langle \xi_{+,2}(\mathrm{Q})\overline{\xi}_{+,2}(\mathrm{Q}')\rangle = -\dfrac{\delta_{\mathrm{Q},\mathrm{Q}'}}{g^*(\theta)},$
- $G_2^{\widetilde{\mathbb{T}}}(\mathrm{Q},\mathrm{Q}') = \langle \xi_{-,2}(\mathrm{Q})\overline{\xi}_{-,2}(\mathrm{Q}')\rangle = -\dfrac{\delta_{\mathrm{Q},\mathrm{Q}'}}{g(\theta)}.$

Proof 5.12 The correlators can be calculated using the inverse rule for 2×2-matrices. The details are carried out in Appendix, Proofs, page 98.

5.3 One-Loop-Approximation

5.3.1 Demanding Self-Consistency

Demanding self-consistency of the previous approximations is equivalent to including the interactions into the action as perturbation again:

$$S_{\mathrm{MF}} \mapsto S_{\mathrm{MF}} + S_{\mathrm{int}}.$$

This action will be approximated in the next step again (this time in the framework of the One-Loop-approximation).

5.3.2 Diagrammatic Language

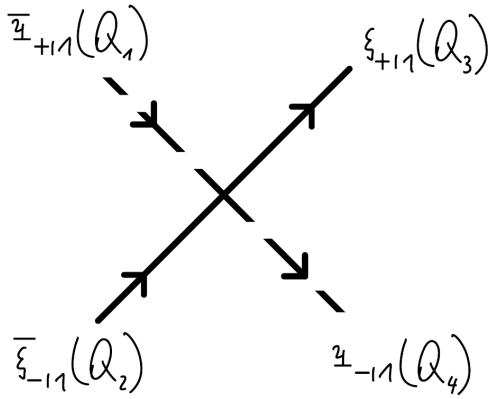
Notation 5.13 Looking at the interacting part of the action in the variables $\{\xi, \overline{\xi}\}$, one quickly realizes why physicists were so receptive to the concept of Feynman-diagrams. For example the term

$$\begin{aligned}
&\frac{2\kappa}{M} \iint\!\!\!\sum_{\mathrm{Q}_1,\ldots,\mathrm{Q}_4} \delta(\mathrm{Q}_1+\mathrm{Q}_2-(\mathrm{Q}_3+\mathrm{Q}_4))\, \overline{\psi}_{+,1}(\mathrm{Q}_1)\overline{\chi}_{-,1}(\mathrm{Q}_2)\overline{\chi}_{+,1}(\mathrm{Q}_3)\overline{\psi}_{-,1}(\mathrm{Q}_4) \\
=&\frac{2\kappa}{M} \iint\!\!\!\sum_{\mathrm{Q}_1,\ldots,\mathrm{Q}_4} \delta(\mathrm{Q}_1+\mathrm{Q}_2-(\mathrm{Q}_3+\mathrm{Q}_4)) \frac{f^*(1,q_1,\theta)f(1,q_4,\theta)}{2^2\prod_{i=1}^4 |f(q_i)|} \left(\overline{\xi}_{+,1}(\mathrm{Q}_1)-\overline{\xi}_{+,2}(\mathrm{Q}_1)\right) \overline{\xi}_{-,1}(\mathrm{Q}_2)\xi_{+,1}(\mathrm{Q}_3) \left(\xi_{-,1}(\mathrm{Q}_4)-\xi_{-,2}(\mathrm{Q}_4)\right)
\end{aligned}$$

where

$$\delta(\mathrm{Q}_1+\mathrm{Q}_2-(\mathrm{Q}_3+\mathrm{Q}_4)) \equiv \delta(\omega_1+\omega_2-(\omega_3+\omega_4))\, \delta_{q_1+q_2-(q_3+q_4)}$$

can be succinctly denoted by the following Feynman-diagram:



Remark 5.13.1 This notation will be crucial when we depict the One-Loop-corrections as Feynman-diagrams in the next subsection.

Remark 5.13.2 The quartic contributions of the action in eq. (3.3) for the exact Lindblad-equation are summarized in the tables in Appendix, page 103.

Remark 5.13.3 Also note that the collectivity of quartic contributions passes the "sanity-check" from Corollary 3.6, which is outlined in Appendix, Remark 8.1.2.

5.3.3 Sorting and Classifying Terms in the One-Loop-Approximation

Resorting to Hypothesis 3.29, we will search for those terms in the One-Loop-Approximation that have divergencies. It will turn out that those divergent corrections only appear at the phase transition (i.e. for $\theta = \pi/4$). Therefore we will agree on the following notation:

Notation All equations in this chapter are to be understood for $\theta = \pi/4$; In this spirit we mostly omit the θ-dependence in the respective equations.

We use Approximation 3.21. To do this, we first introduce the inverse Green's-function

$$S_{\mathrm{MF}}[\xi, \overline{\xi}] \equiv \sum\!\!\!\!\!\!\int_{\alpha,\alpha'} \overline{\xi}(\alpha)\, G_0^{-1}(\alpha, \alpha')\xi(\alpha'),$$

where $\alpha', \alpha = (\omega, q, b, \nu) \in \mathbb{R} \times \mathcal{Q} \times \{1,2\} \times \{+,-\}$. Looking at eq. (3.16) we introduce the symbol

$$\frac{\delta^2}{\delta\overline{\xi}\,\delta\xi} S_{\mathrm{int}}[\xi, \overline{\xi}] \equiv V[\xi, \overline{\xi}] \equiv V.$$

For reasons that will become clear later, we are interested in the "particle-hole-terms" in this expansion. The trace of this term is then evaluated by stepwise "matrix-multiplication", i.e.

$$\begin{aligned} \mathrm{Tr}\left[G_0\, V\, G_0\, V\right] &= \sum\!\!\!\!\!\!\int_{\alpha} \left(G_0\, V\, G_0\, V\right)(\alpha, \alpha) \\ &= \sum\!\!\!\!\!\!\int_{\alpha,\beta,\gamma,\epsilon} G_0(\alpha, \beta)\, V(\beta, \gamma)\, G_0(\gamma, \epsilon)\, V(\epsilon, \alpha) \\ &= \dots \end{aligned} \tag{5.2}$$

After matrix-multiplication and performing the trace (with respect to the energy bands) one is left with a series of terms in sums of the type

$$\sum\!\!\!\!\!\!\int_{\alpha^{(1)},\beta^{(1)},\gamma^{(1)},\epsilon^{(1)}} (\dots),$$

with $\alpha^{(1)}, \beta^{(1)}, \gamma^{(1)}, \epsilon^{(1)} \in \mathbb{R} \times \mathcal{Q} \times \{+,-\}$. Every resulting term becomes further evaluated by matrix-multiplication and performing the trace (with respect to the branches). The resulting terms (about 5000 in total), which can be sorted according to their monomials, are examined below.

Theorem 5.14 All summands on the r.h.s. of eq. (5.2) have the form

$$\pm\frac{2\kappa}{M} \sum\!\!\!\!\!\!\int_{\mathrm{Q}_1,\dots,\mathrm{Q}_4} \delta(\mathrm{Q}_1 + \mathrm{Q}_2 - (\mathrm{Q}_3 + \mathrm{Q}_4)) \cdot \Sigma(\tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4) \cdot \mathrm{L}(\mathrm{Q}_i, \mathrm{Q}_j, \mathrm{GF}_1, \mathrm{GF}_2, n_1, n_2, n_3, n_4) \cdot \overline{\xi}_{I_1}(\mathrm{Q}_1)\overline{\xi}_{I_2}(\mathrm{Q}_2)\xi_{I_3}(\mathrm{Q}_3)\xi_{I_4}(\mathrm{Q}_4)$$

with functions L (as "[L]oop")

$$\begin{aligned} &\mathrm{L} \equiv \mathrm{L}(\mathrm{Q}_i, \mathrm{Q}_j, \mathrm{GF}_1, \mathrm{GF}_2, n_1, n_2, n_3, n_4) \\ &= \frac{2\kappa}{M} \sum\!\!\!\!\!\!\int_{\mathrm{Q}_\alpha,\mathrm{Q}_\epsilon} \delta(\mathrm{Q}_i + \mathrm{Q}_\epsilon - (\mathrm{Q}_j + \mathrm{Q}_\alpha))\, \mathrm{GF}_1(\mathrm{Q}_\alpha)\, \mathrm{GF}_2(\mathrm{Q}_\epsilon)\, \frac{f^*(n_1, q_\alpha)^{\delta(n_1\neq 0)} f(n_2, q_\alpha)^{\delta(n_2\neq 0)} f^*(n_3, q_\epsilon)^{\delta(n_3\neq 0)} f(n_4, q_\epsilon)^{\delta(n_4\neq 0)}}{2^2 |f(q_\epsilon)|^2 |f(q_\alpha)|^2} \end{aligned}$$

and

$$\Sigma(\tilde{n}_1,\tilde{n}_2,\tilde{n}_3,\tilde{n}_4)=\frac{f^*(\tilde{n}_1,q_1)^{\delta(\tilde{n}_1\neq 0)}f^*(\tilde{n}_2,q_2)^{\delta(\tilde{n}_2\neq 0)}f(\tilde{n}_3,q_3)^{\delta(\tilde{n}_3\neq 0)}f(\tilde{n}_4,q_4)^{\delta(\tilde{n}_4\neq 0)}}{2^2\prod_{j=1}^4|f(q_j)|},$$

where $i,j\in\{1,\dots,4\}$, $(\tilde{n}_1,\tilde{n}_2,\tilde{n}_3,\tilde{n}_4)\in\{0,1,2\}^4$, $\mathrm{GF}_{1,2}\in\bigcup_{b=1,2}\{G_b^{\mathbb{T}},G_b^{\widetilde{\mathbb{T}}},G_b^<,G_b^>\}$ and

$$\sum_{j=1}^4\delta(n_j\neq 0)+\delta(\tilde{n}_j\neq 0)=4\qquad\text{with}\qquad\delta(A\neq B)=\begin{cases}1 & \text{for } A\neq B\\ 0 & \text{for } A=B.\end{cases}$$

Proof 5.14 Follows by insertion in the formula and staring at the resulting expressions. Alternatively one can also conclude this with the concrete form of the interaction: each quadratic functional derivative gives either zero, one or two multiplicative f-valued contributions in L. Correspondingly, in Σ there are then two, one or zero multiplicative f-valued contributions.[4]

To proceed now, we first notice the factorization of the loops as

$$\begin{aligned}\mathrm{L}(\mathrm{Q}_i,\mathrm{Q}_j,\mathrm{GF}_1,\mathrm{GF}_2,n_1,n_2,n_3,n_4)&=\frac{2\kappa}{M}\frac{1}{2^2}\sum_q\mathscr{F}(q,\Delta q,n_1,\dots,n_4)\int\frac{\mathrm{d}\omega}{2\pi}\mathrm{GF}_1(\mathrm{Q})\,\mathrm{GF}_2(\mathrm{Q}+\Delta\,\mathrm{Q})\\&\equiv\frac{2\kappa}{M}\frac{1}{2^2}\sum_q\mathscr{F}(q,\Delta q,n_1,\dots,n_4)\cdot\mathcal{J}(\mathrm{GF}_1,\mathrm{GF}_2,\Delta\,\mathrm{Q}),\end{aligned}\tag{5.3}$$

with the factor

$$\mathscr{F}(q,\Delta q,n_1,\dots,n_4)=\frac{f^*(n_1,q)^{\delta(n_1\neq 0)}f(n_2,q)^{\delta(n_2\neq 0)}f^*(n_3,q+\Delta q)^{\delta(n_3\neq 0)}f(n_4,q+\Delta q)^{\delta(n_4\neq 0)}}{|f(q)|^2|f(q+\Delta q)|^2},\tag{5.4}$$

and the abbreviation

$$\mathrm{Q}_j-\mathrm{Q}_i\equiv\Delta\,\mathrm{Q}\equiv(\Delta\omega,\Delta q).$$

We divide ("and conquer") the problem, by considering the two factors in eq. (5.3) separately.

- **The Factor $\mathscr{F}$.**

  We consider (in "wise foresight") the sets $\mathcal{N}_{1,2,3}$, where

  $$\begin{aligned}\mathcal{N}_1&=\{(1,1,1,1),(1,1,2,2),(2,2,1,1),(2,2,2,2)\},\\ \mathcal{N}_2&=\{(1,1,1,2),(1,1,2,1),(1,2,1,1),(2,1,1,1),(2,2,1,2),(2,2,2,1),(1,2,2,2),(2,1,2,2)\},\\ \mathcal{N}_3&=\{(1,2,1,2),(1,2,2,1),(2,1,1,2),(2,1,2,1)\}.\end{aligned}$$

[4] With this consideration, and later with Lemma 5.15, it will also become clear, why only particle-hole-diagrams, and not particle-particle-diagrams diverge: Functional derivatives of the type $\frac{\delta^2}{\delta\xi\delta\xi}$ and $\frac{\delta^2}{\delta\bar{\xi}\delta\bar{\xi}}$ can create either zero or one multiplicative f-valued contributions in the particle-particle-diagram-version of L. These are not enough f-valued contributions to compensate the factor of $1/|f(q_e)|^2|f(q_\alpha)|^2$ in L that *cures* divergencies, (as will become clear later)!

Lemma 5.15 It holds

$$(n_1, \dots, n_4) \equiv n \in \bigcup_{i=1}^{3} \mathcal{N}_i \quad \Rightarrow \quad \mathscr{F}(q, \Delta q, n) \equiv 1.$$

Proof 5.15 Follows immediately with the identities in Remark 4.10.1.

- **The Frequency-Integral $\mathcal{J}$.**

  By the continuum-approximation, we can treat the frequency-integral with the residue theorem. Since GF_1 and GF_2 are respectively linear combinations of $1/g$ and $1/g^*$ (cf. Corollary 5.12), it is sufficient to compute a single integral.

  Lemma 5.16 It holds

$$\int \frac{\mathrm{d}\omega}{2\pi} \frac{1}{g(\omega, q)} \cdot \frac{1}{g^*(\omega + \Delta\omega, q + \Delta q)} = \frac{1}{-\mathrm{i}\Delta\omega + \mathrm{K}(q) + \mathrm{K}(q + \Delta q)}.$$

  Proof 5.16 By closing the contour in the upper half plane. See Appendix, Proofs, page 99 for details.

Notation 5.17 We introduce the abbreviation D (as [D]ivergence)

$$D(\mathrm{Q}_i, \mathrm{Q}_j) \equiv D(\Delta\,\mathrm{Q}) = \frac{2\kappa}{M} \frac{1}{2^2} \sum_q \frac{1}{\mathrm{i}\Delta\omega + (\mathrm{K}(q) + \mathrm{K}(q + \Delta q))}. \tag{5.5}$$

Lemma 5.18 It immediately follows

$$\mathrm{Re}\, D(\Delta\,\mathrm{Q}) = \frac{2\kappa}{M} \frac{1}{2^2} \sum_q \frac{\mathrm{K}(q) + \mathrm{K}(q + \Delta q)}{(\Delta\omega)^2 + (\mathrm{K}(q) + \mathrm{K}(q + \Delta q))^2}, \qquad \mathrm{Re}\, D(\Delta\,\mathrm{Q}) = \mathrm{Re}\, D(-\Delta\,\mathrm{Q}),$$

$$\mathrm{Im}\, D(\Delta\,\mathrm{Q}) = \frac{2\kappa}{M} \frac{1}{2^2} \sum_q \frac{-\Delta\omega}{(\Delta\omega)^2 + (\mathrm{K}(q) + \mathrm{K}(q + \Delta q))^2}, \qquad \mathrm{Im}\, D(\Delta\,\mathrm{Q}) = -\,\mathrm{Im}\, D(-\Delta\,\mathrm{Q}),$$

$$D(-\Delta\,\mathrm{Q}) = \mathrm{Re}\, D(-\Delta\,\mathrm{Q}) + \mathrm{i}\,\mathrm{Im}\, D(-\Delta\,\mathrm{Q}) = \mathrm{Re}\, D(\Delta\,\mathrm{Q}) - \mathrm{i}\,\mathrm{Im}\, D(\Delta\,\mathrm{Q}) = D^*(\Delta\,\mathrm{Q}).$$

and at the phase transition, i.e. at $\theta = \pi/4$, one has an infrared-divergence

$$D(\Delta\,\mathrm{Q} \to 0) \quad \propto \quad \text{const.} \quad + \quad \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \frac{\mathrm{d}\delta}{2\pi} \frac{1}{\delta^2} \quad \to \quad \infty.$$

Proof 5.18 See Appendix, Proofs, page 99.

Lemma 5.19 For $n \equiv (n_1, \ldots, n_4) \in \bigcup_{i=1}^{3} \mathcal{N}_i$ one obtains the mapping

$$(\mathrm{GF}_1, \mathrm{GF}_2) \mapsto \mathrm{L}(\mathrm{Q}_i, \mathrm{Q}_j, \mathrm{GF}_1, \mathrm{GF}_2, n) = c_1 D(\Delta\,\mathrm{Q}) + c_2 D^*(\Delta\,\mathrm{Q}),$$

where

$$c_{1,2} \in \{-1, 0, +1\}.$$

This provides the table in Figure 5.2.

Proof 5.19 Follows immediately with Lemma 5.15 and Lemma 5.16, and subsequent expansion.

GF 1 \ GF 2	$G_1^<$	$G_1^>$	$G_1^\top$	$G_1^{\widetilde{\top}}$	$G_2^<$	$G_2^>$	$G_2^\top$	$G_2^{\widetilde{\top}}$
$G_1^<$	0	0	0	0	0	0	0	0
$G_1^>$	0	$2\mathrm{Re}D$	D	D^*	$-2\mathrm{Re}D$	0	$-D^*$	$-D$
$G_1^\top$	0	D^*	0	D^*	$-D^*$	0	$-D^*$	0
$G_1^{\widetilde{\top}}$	0	D	D	0	$-D$	0	0	$-D$
$G_2^<$	0	$-2\mathrm{Re}D$	$-D$	$-D^*$	$2\mathrm{Re}D$	0	D^*	D
$G_2^>$	0	0	0	0	0	0	0	0
$G_2^\top$	0	$-D$	$-D$	0	D	0	0	D
$G_2^{\widetilde{\top}}$	0	$-D^*$	0	$-D^*$	D^*	0	D^*	0

Figure 5.2: Image of the function
$(\mathrm{GF}_1, \mathrm{GF}_2) \mapsto c_1 D(\Delta\,\mathrm{Q}) + c_2 D^*(\Delta\,\mathrm{Q})$
with $D \equiv D(\Delta\,\mathrm{Q})$.

Theorem 5.20 The only terms in the One-Loop-approximation that give divergent corrections are those corresponding to the sets $\mathcal{N}_{1,2,3}$.

Proof 5.20 See Appendix, Proofs, page 101.

Theorem 5.21 Writing

$$\begin{aligned}\Gamma_{1-\mathrm{Loop}}[\xi,\bar{\xi}] &= \Gamma_{\mathrm{Tree}}[\xi,\bar{\xi}] + \Gamma_{\mathrm{Tr}\log}[\xi,\bar{\xi}] \\ &= \Gamma_{\mathrm{Tree}}[\xi,\bar{\xi}] + \Gamma_{\infty}[\xi,\bar{\xi}] + \Gamma_{\mathrm{Rest}}[\xi,\bar{\xi}] \\ &= \Gamma_{\infty}[\xi,\bar{\xi}] \quad + \Gamma_{\widetilde{\mathrm{Rest}}}[\xi,\bar{\xi}],\end{aligned}$$

with the obvious phraseology[5] gives with the abbreviations

$$\Xi \equiv \Xi(\mathrm{Q}_1,\ldots,\mathrm{Q}_4) \equiv \frac{2\kappa}{M}\frac{\delta(\mathrm{Q}_1+\mathrm{Q}_2-(\mathrm{Q}_3+\mathrm{Q}_4))}{2^2\prod_{i=1}^{4}|f(q_i)|}$$

the following table for the divergent terms:

Index	Vertex-Function		
17	$\gamma_{\infty}{}^{1212}_{+++-}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$4\,\Xi\, D(\mathrm{Q}_2,\mathrm{Q}_4)$
25	$\gamma_{\infty}{}^{2222}_{++--}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$-8\,\Xi\,\mathrm{Re}\, D(\mathrm{Q}_2,\mathrm{Q}_4)$
28	$\gamma_{\infty}{}^{2222}_{+++-}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$4\,\Xi\, D(\mathrm{Q}_2,\mathrm{Q}_4)$
30	$\gamma_{\infty}{}^{1111}_{--++}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$-8\,\Xi\,\mathrm{Re}\, D(\mathrm{Q}_2,\mathrm{Q}_4)$
36	$\gamma_{\infty}{}^{1111}_{--+-}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$4\,\Xi\, D^*(\mathrm{Q}_2,\mathrm{Q}_4)$
47	$\gamma_{\infty}{}^{1212}_{--+-}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$4\,\Xi\, D^*(\mathrm{Q}_2,\mathrm{Q}_4)$
60	$\gamma_{\infty}{}^{1111}_{+-++}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$4\,\Xi\, D(\mathrm{Q}_2,\mathrm{Q}_4)$
85	$\gamma_{\infty}{}^{2222}_{+---}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$4\,\Xi\, D^*(\mathrm{Q}_2,\mathrm{Q}_4)$
91	$\gamma_{\infty}{}^{1212}_{-+++}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$4\,\Xi\, D^*(\mathrm{Q}_2,\mathrm{Q}_4)$
94	$\gamma_{\infty}{}^{1212}_{-+--}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$4\,\Xi\, D(\mathrm{Q}_2,\mathrm{Q}_4)$
97	$\gamma_{\infty}{}^{1212}_{-++-}(\mathrm{Q}_1,\ldots,\mathrm{Q}_4)$	=	$-16\,\Xi\,\mathrm{Re}\, D(\mathrm{Q}_2,\mathrm{Q}_4)$

Proof 5.21 Follows by numerical analysis. (Approximately 2000 lines of code in a Java-program.) Available on request.[6]

Remark 5.21.1 The numbers of the indices originate from the tables beginning on page 103 in the appendix. There the divergent vertex-corrections of the One-Loop-approximation are again contrasted to the vertex-functions of the action itself.

[5] Γ_{∞} possesses divergent vertex-functions γ_{∞}.
[6] Sebastian.Kalhoefer@Gmail.com

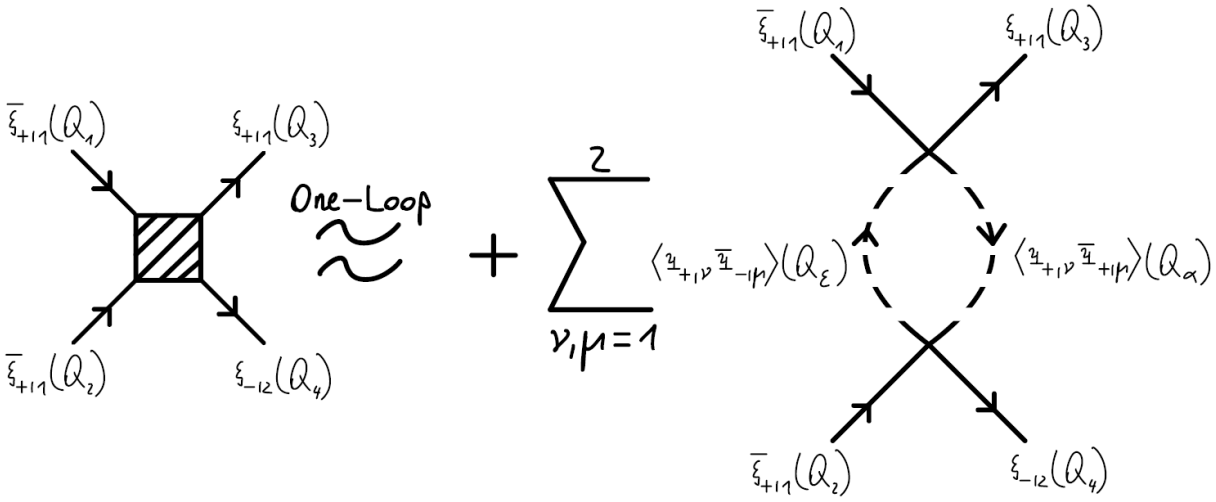
Pictographic Representation of the Divergencies

Fact: All the divergencies obey a diagrammatic interpretation as particle-hole-diagrams with loops generated by correlators that correspond to the expectation values of Graßmann-numbers from Wannier-states.

For example[7], the approximation of the vertex-function

$$\gamma^{1212}_{+++-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4) \approx \gamma_{\infty}{}^{1212}_{+++-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$$

can be represented and decomposed as



where the factor of $1/2$ from the trace-log-expansion has been eliminated by incorporating the symmetry $\mathrm{GF}_1 \leftrightarrow \mathrm{GF}_2$ and non-divergent diagrams have been neglected. Similar decompositions can be written down for all other divergent vertex-functions, which can be seen from Lemma 5.15.

5.3.4 "Sanity-Check"

By Lemma 3.6 the corrections should truncate each other if we make the substitution $(+,-) \mapsto (+,+)$ for the branches. Indeed we obtain ...

For the 1111**-Quadruples:**

$$\left(\gamma_{\infty,30} + \gamma_{\infty,36} + \gamma_{\infty,60}\right)_{|_{(+,-)\mapsto(+,+)}} = 4\,\Xi\,(-2\ \mathrm{Re}\,D\ +\ D^*\ +\ D) = 0.$$

For the 1212**-Quadruples:**

$$\left(\gamma_{\infty,17} + \gamma_{\infty,47} + \gamma_{\infty,91} + \gamma_{\infty,94} + \gamma_{\infty,97}\right)_{|_{(+,-)\mapsto(+,+)}} = 4\,\Xi\,(D\ +\ D^*\ +\ D^*\ +\ D\ -\ 4\,\mathrm{Re}\,D) = 0.$$

For the 2222**-Quadruples:**

$$\left(\gamma_{\infty,25} + \gamma_{\infty,28} + \gamma_{\infty,85}\right)_{|_{(+,-)\mapsto(+,+)}} = 4\,\Xi\,(-2\ \mathrm{Re}\,D\ +\ D\ +\ D^*) = 0.$$

which is a strong indication that we have not miscalculated along the way.

[7] Note that we have omitted the Graßmann-quadrupel $\overline{\xi}_{+,1}(\mathrm{Q}_1)\,\overline{\xi}_{+,2}(\mathrm{Q}_2)\,\xi_{+,1}(\mathrm{Q}_3)\,\xi_{-,2}(\mathrm{Q}_4)$ in the first equation for notational simplicity. The two equations are equivalent to each other, but differ by "multiplication" of the Graßmann-quadrupel.

6 The Renormalization Group and its Flow

In this chapter we will concretely apply the formalism we have introduced in section 3.3.3. We enumerate the quadratic vertex-functions according to

$$\gamma^{\Lambda}_{2,0}(Q_1, Q_2), \quad \gamma^{\Lambda}_{2,1}(Q_1, Q_2), \quad \dots, \quad \gamma^{\Lambda}_{2,15}(Q_1, Q_2),$$

and the quartic vertex-functions according to[1]

$$\gamma^{\Lambda}_{4,0}(Q_1, Q_2, Q_3, Q_4), \quad \gamma^{\Lambda}_{4,1}(Q_1, Q_2, Q_3, Q_4), \quad \dots, \quad \gamma^{\Lambda}_{4,99}(Q_1, Q_2, Q_3, Q_4).$$

Then for $\bullet \in \{0, \dots, 99\}$, the flow equations in eq. (3.21) can be written as[2]

$$\frac{\mathrm{d}}{\mathrm{d}\Lambda}\gamma^{\Lambda}_{4,\bullet}(Q_1, \dots, Q_4) = \mathscr{F}_{\bullet}[\gamma^{\Lambda}_{2,0}, \dots, \gamma^{\Lambda}_{2,15}, \gamma^{\Lambda}_{4,0}, \dots, \gamma^{\Lambda}_{4,99}]. \tag{6.1}$$

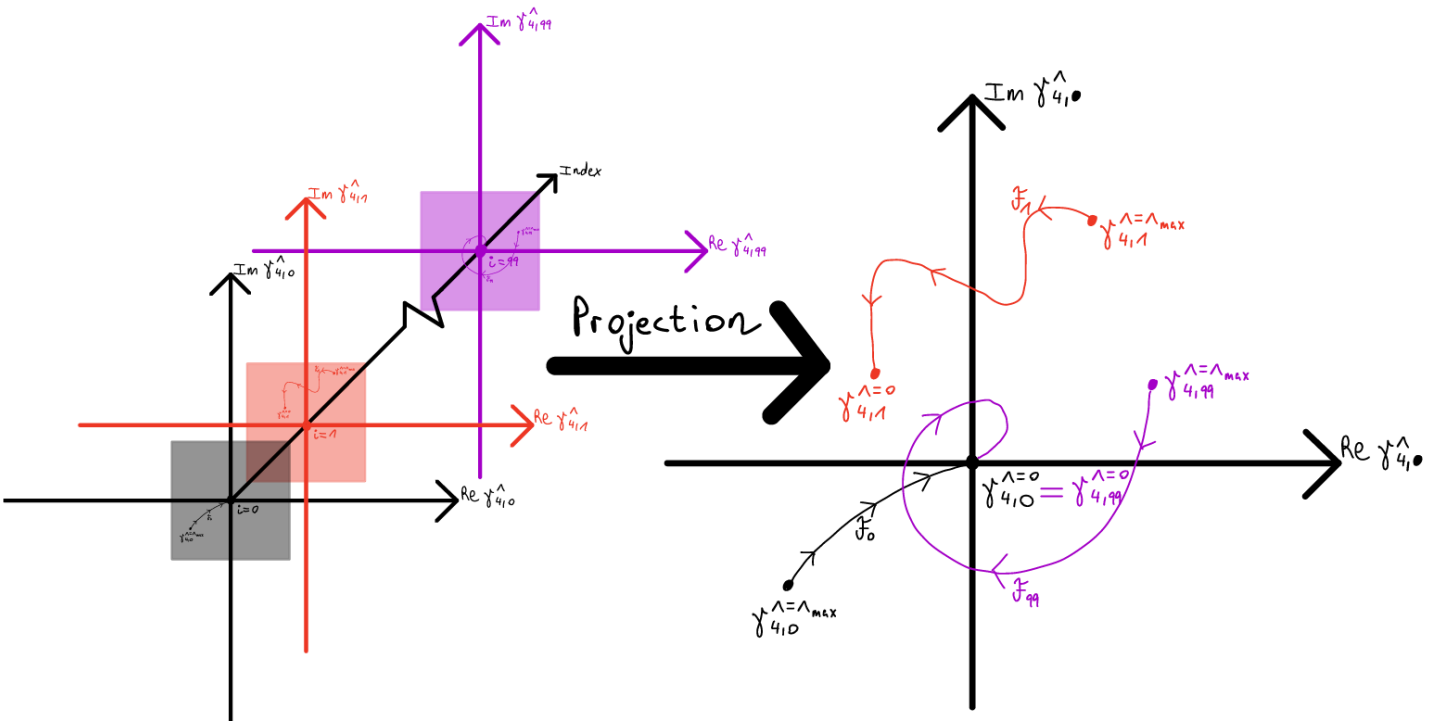


Figure 6.1: Renormalization concept for fixed momentum configuration $(Q_1, \dots, Q_4)$: The different couplings "flow" when Λ changes. Since all quartic couplings have the same dimension, one can (at least for a fixed momentum-configuration $\gamma^{\Lambda}_{4,\bullet} \equiv \gamma^{\Lambda}_{4,\bullet}(Q_1, \dots, Q_4)$) represent the flow in the complex plane.

[1] See also the table in Appendix, page 103.

[2] Note that we are using the symbol $\mathscr{F}_{\bullet}$ instead of $\beta_{\bullet}$ (from section 3.3.4), since the beta-functions have been introduced for dimension*less* couplings, while the vertex-functions might carry some dimension.

6.1 Wetterich-Equation, Regulator, and Flow-Equations

The divergencies of the one-loop-approximation arise from a singularity for a certain momentum-mode ($q = \pi/a$) which arises in the loop-contributions in the $\mathrm{Q}_1 = \mathrm{Q}_2 = \mathrm{Q}_3 = \mathrm{Q}_4$-sector. In section 3.3.3 we therefore choose

$$f(\alpha) \equiv f(\omega, q, b, \nu) := \mathrm{K}(q, \theta), \qquad \gamma_\bullet^\Lambda(\mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3, \mathrm{Q}_4) \overset{\text{Ansatz}}{\approx} \delta(\mathrm{Q}_1 + \mathrm{Q}_2 - (\mathrm{Q}_3 + \mathrm{Q}_4)) \cdot \gamma_\bullet(\Lambda),$$

where $\bullet \in \{0, 1, \ldots, 99\}$ and consider the flows of the couplings in this sector. This procedure is very similar to the so-called "two-patch approximation" in [6] section 4.4. In the spirit of section 3.3.3 we obtain for the Λ-dependent Green's-function

$$G_{0,\Lambda}^{-1}(\alpha, \alpha') := \left(\mathrm{Mat}[\theta_{\Lambda,\epsilon}] \cdot G_0\right)^{-1}(\alpha, \alpha'),$$

where $\mathrm{Mat}[\theta_{\Lambda,\epsilon}](\alpha, \alpha')$ denotes a diagonal, regularized Heaviside-matrix

$$\mathrm{Mat}[\theta_{\Lambda,\epsilon}](\alpha, \alpha') := \delta(\alpha - \alpha')\, \theta_\epsilon(\mathrm{K}(q, \theta) - \Lambda), \tag{6.2}$$

and, as before, $\alpha', \alpha = (\omega, q, b, \nu) \in \mathbb{R} \times \mathcal{Q} \times \{1, 2\} \times \{+, -\}$.

Theorem 6.1 For the r.h.s. of eq. (3.24) one obtains dominant contributions from terms of the shape

$$\pm \oint\!\!\!\!\!\!\sum_{\mathrm{Q}_1, \ldots, \mathrm{Q}_4} \delta(\mathrm{Q}_1 + \mathrm{Q}_2 - (\mathrm{Q}_3 + \mathrm{Q}_4)) \cdot \gamma_k(\Lambda)\, \gamma_l(\Lambda) \cdot \mathrm{L}_{\Lambda,\epsilon} \cdot \overline{\xi}_{I_1}(\mathrm{Q}_1) \overline{\xi}_{I_2}(\mathrm{Q}_2) \xi_{I_3}(\mathrm{Q}_3) \xi_{I_4}(\mathrm{Q}_4),$$

with functions $\mathrm{L}_{\Lambda,\epsilon}$ (as „[L]oop")

$$\begin{aligned}
\mathrm{L}_{\Lambda,\epsilon} &\equiv \mathrm{L}(\mathrm{Q}_i, \mathrm{Q}_j, \mathrm{mat}[\delta_{\Lambda,\epsilon}] \cdot \mathrm{GF}_1, \mathrm{mat}[\theta_{\Lambda,\epsilon}] \cdot \mathrm{GF}_2) \\
&= \oint\!\!\!\!\!\!\sum_{\mathrm{Q}_\alpha, \mathrm{Q}_\epsilon} \delta(\mathrm{Q}_i + \mathrm{Q}_\epsilon - (\mathrm{Q}_j + \mathrm{Q}_\alpha)) \left(\mathrm{mat}[\delta_{\Lambda,\epsilon}] \cdot \mathrm{GF}_1\right)(\mathrm{Q}_\alpha) \left(\mathrm{mat}[\theta_{\Lambda,\epsilon}] \cdot \mathrm{GF}_2\right)(\mathrm{Q}_\epsilon) \\
&= \sum_{\mathrm{Q}} \left(\mathrm{mat}[\delta_{\Lambda,\epsilon}] \cdot \mathrm{GF}_1\right)(\mathrm{Q}) \left(\mathrm{mat}[\theta_{\Lambda,\epsilon}] \cdot \mathrm{GF}_2\right)(\mathrm{Q} + \Delta\,\mathrm{Q}),
\end{aligned}$$

where $i, j \in \{1, \ldots, 4\}$, $\mathrm{GF}_{1,2} \in \bigcup_{b=1,2} \{G_b^{\mathbb{T}}, G_b^{\widetilde{\mathbb{T}}}, G_b^<, G_b^>\}$, $k, l \in \{0, 1, \ldots, 99\}$, $I_{1,\ldots,4} \in \{+, -\} \times \{1, 2\}$ and

$$\mathrm{mat}[\delta_{\Lambda,\epsilon}](\mathrm{Q}, \mathrm{Q}') = \delta(\omega - \omega')\, \delta_\epsilon(\mathrm{K}(q, \theta) - \Lambda),$$

leading to regularized divergencies $D_{\Lambda,\epsilon}$.

Proof 6.1 Follows directly through the formal correspondence mentioned in Remark 3.33.1: The terms in eq. (5.2) translate naturally to the terms of the flow equations in eq. (3.21)

$$\mathrm{Tr}\left[G_0\, V\, G_0\, V\right] \mapsto \mathrm{Tr}\left[\partial_\Lambda G_{0,\Lambda} \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_{\mathrm{int},\Lambda}[\phi, \overline{\phi}] \cdot G_{0,\Lambda} \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_{\mathrm{int},\Lambda}[\phi, \overline{\phi}]\right].$$

Now, we follow the logic in the last section and use Hypothesis 3.29:

Theorem 6.2 In the $Q_1 = Q_2 = Q_3 = Q_4$-sector one obtains the flow equations

$$\begin{cases} \frac{d}{d\Lambda}\gamma^{1212}_{+++-}(\Lambda) \approx \gamma^{1212}_{++++}(\Lambda) \cdot \left(4\, D_{\Lambda,\epsilon}\right) \cdot \gamma^{2222}_{+-+-}(\Lambda), \\ \frac{d}{d\Lambda}\gamma^{1212}_{--+-}(\Lambda) \approx \gamma^{1212}_{----}(\Lambda) \cdot \left(4\, D_{\Lambda,\epsilon}\right) \cdot \gamma^{1111}_{+-+-}(\Lambda), \\ \frac{d}{d\Lambda}\gamma^{1212}_{-+++}(\Lambda) \approx \gamma^{1212}_{++++}(\Lambda) \cdot \left(4\, D_{\Lambda,\epsilon}\right) \cdot \gamma^{1111}_{+-+-}(\Lambda), \\ \frac{d}{d\Lambda}\gamma^{1212}_{-+--}(\Lambda) \approx \gamma^{1212}_{----}(\Lambda) \cdot \left(4\, D_{\Lambda,\epsilon}\right) \cdot \gamma^{2222}_{+-+-}(\Lambda), \\ \frac{d}{d\Lambda}\gamma^{1212}_{-++-}(\Lambda) \approx \gamma^{1212}_{-++-}(\Lambda) \cdot \left(-8\, D_{\Lambda,\epsilon}\right) \cdot \gamma^{1111}_{+-+-}(\Lambda) \quad + \quad \gamma^{1212}_{-++-}(\Lambda) \cdot \left(-8\, D_{\Lambda,\epsilon}\right) \cdot \gamma^{2222}_{+-+-}(\Lambda). \end{cases} \tag{6.3}$$

Proof 6.2 Follows by numerical analysis. Available on request.[3]

Corollary 6.3 With symmetry considerations one concludes that one can solve the system of differential equations in Theorem 6.2 if and only if one can solve the following system:

$$\left.\begin{aligned} \frac{d}{d\Lambda}\gamma^{1212}_{-+++}(\Lambda) &\approx \gamma^{1212}_{++++}(\Lambda) \cdot \left(4\, D_{\Lambda,\epsilon}\right) \cdot \gamma^{1111}_{+-+-}(\Lambda), \\ \frac{d}{d\Lambda}\gamma^{1212}_{-++-}(\Lambda) &\approx \gamma^{1212}_{-++-}(\Lambda) \cdot \left(-16\, D_{\Lambda,\epsilon}\right) \cdot \gamma^{1111}_{+-+-}(\Lambda) \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} \frac{d}{d\Lambda}\gamma_{91}(\Lambda) &\approx \gamma_{11}(\Lambda) \cdot \left(4\, D_{\Lambda,\epsilon}\right) \cdot \gamma_{66}(\Lambda) \\ \frac{d}{d\Lambda}\gamma_{97}(\Lambda) &\approx \gamma_{97}(\Lambda) \cdot \left(-16\, D_{\Lambda,\epsilon}\right) \cdot \gamma_{66}(\Lambda) \end{aligned}\right. .$$

Proof 6.3 The terms $\gamma_{66}(\Lambda)$ and $\gamma_{88}(\Lambda)$, $\gamma_{11}(\Lambda)$ and $\gamma_{44}(\Lambda)$ obviously coincide in the limit $\Lambda \to \infty$ which can be read off from the tables starting on page 103 using the initial conditions from eq. (3.23). The first four equations are therefore equivalent, and in the last equation one can simply collect the contributions from vertices that flow in the same way.

Therefore, the flow-equations can be reduced to the set of equations for the dimensional couplings

$$\begin{cases} \frac{d}{d\Lambda}\gamma_{11}(\Lambda) \approx 0 \\ \frac{d}{d\Lambda}\gamma_{66}(\Lambda) \approx 0 \\ \frac{d}{d\Lambda}\gamma_{91}(\Lambda) \approx \gamma_{11}(\Lambda) \cdot \left(4\, D_{\Lambda,\epsilon}\right) \cdot \gamma_{66}(\Lambda) \\ \frac{d}{d\Lambda}\gamma_{97}(\Lambda) \approx \gamma_{97}(\Lambda) \cdot \left(-16\, D_{\Lambda,\epsilon}\right) \cdot \gamma_{66}(\Lambda) \end{cases} \tag{6.4}$$

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6.2 Critical Exponents

6.2.1 Dimensional Analysis

Before we look for fixed points of the flow-equations we have to do a dimensional analysis. For the introduction of dimension*less* vertex-functions, it is common to rewrite the action in continuum-notation. Looking at the mean field action from Approximation 5.11

$$S_{\mathrm{MF}}[\xi,\bar{\xi}] \approx \int \frac{\mathrm{d}\omega}{2\pi} \sum_q \begin{pmatrix} \bar{\xi}_{+,1} \\ \bar{\xi}_{-,1} \\ \bar{\xi}_{+,2} \\ \bar{\xi}_{-,2} \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} -\mathrm{i}\omega - \mathrm{K}(q,\theta) & 0 & & \\ 2\,\mathrm{K}(q,\theta) & \mathrm{i}\omega - \mathrm{K}(q,\theta) & & \\ & & -\mathrm{i}\omega + \mathrm{K}(q,\theta) & -2\,\mathrm{K}(q,\theta) \\ & & 0 & \mathrm{i}\omega + \mathrm{K}(q,\theta) \end{pmatrix} \cdot \begin{pmatrix} \xi_{+,1} \\ \xi_{-,1} \\ \xi_{+,2} \\ \xi_{-,2} \end{pmatrix},$$

we see that we can transition to the continuum-version of the action by rescaling[4] Graßmann-fields as

$$\xi \mapsto \frac{\xi}{\sqrt{L}}, \qquad \bar{\xi} \mapsto \frac{\bar{\xi}}{\sqrt{L}},$$

where $L \equiv M\,a$, yielding

$$\begin{aligned} S_{\mathrm{MF}}[\xi,\bar{\xi}] &\approx \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}q}{2\pi} \begin{pmatrix} \bar{\xi}_{+,1} \\ \bar{\xi}_{-,1} \\ \bar{\xi}_{+,2} \\ \bar{\xi}_{-,2} \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} -\mathrm{i}\omega - \mathrm{K}(q,\theta) & 0 & & \\ 2\,\mathrm{K}(q,\theta) & \mathrm{i}\omega - \mathrm{K}(q,\theta) & & \\ & & -\mathrm{i}\omega + \mathrm{K}(q,\theta) & -2\,\mathrm{K}(q,\theta) \\ & & 0 & \mathrm{i}\omega + \mathrm{K}(q,\theta) \end{pmatrix} \cdot \begin{pmatrix} \xi_{+,1} \\ \xi_{-,1} \\ \xi_{+,2} \\ \xi_{-,2} \end{pmatrix} \\ &\equiv \int \frac{\mathrm{d}\,\mathrm{Q}}{(2\pi)^2} \begin{pmatrix} \bar{\xi}_{+,1} \\ \bar{\xi}_{-,1} \\ \bar{\xi}_{+,2} \\ \bar{\xi}_{-,2} \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} -\mathrm{i}\omega - \mathrm{K}(q,\theta) & 0 & & \\ 2\,\mathrm{K}(q,\theta) & \mathrm{i}\omega - \mathrm{K}(q,\theta) & & \\ & & -\mathrm{i}\omega + \mathrm{K}(q,\theta) & -2\,\mathrm{K}(q,\theta) \\ & & 0 & \mathrm{i}\omega + \mathrm{K}(q,\theta) \end{pmatrix} \cdot \begin{pmatrix} \xi_{+,1} \\ \xi_{-,1} \\ \xi_{+,2} \\ \xi_{-,2} \end{pmatrix}. \end{aligned}$$

Now as

$$\mathrm{K}(q+\Delta q,\theta) = \kappa\,(1+\sin(2\theta)\cos((q+\Delta q)a)) \sim (\Delta q a)^2,$$

in the vicinity of the critical point $(\theta, q) = (\pi/4, \pi/a)$, we conclude that

$$\omega \sim k^2, \qquad \text{with} \quad q \sim k.$$

Therefore we have $\mathrm{d}\,\mathrm{Q} \sim k^3$. Introducing now

$$\bar{\xi} \sim \xi \sim k^s$$

gives the first equation

[4] The integration measure is implicitly adapted and the momentum-integral extends over the first Brillouin-zone.

$$0 = 3 + 2 + 2 \cdot s \quad \Longleftrightarrow \quad s = -\frac{5}{2}.$$

To proceed, we remember the series-expansion from eq. (3.7)

$$\begin{aligned}\Gamma_\Lambda[\xi,\bar{\xi}] &\equiv \sum_{n,m=0}^{\infty} \oint_{\substack{x_{i_1}<\ldots<x_{i_n} \\ y_{j_1}<y_{j_2}<\ldots<y_{j_{m-1}}<y_{j_m}}} \gamma^\Lambda(x_{i_1},\ldots,x_{i_n},y_{j_1},\ldots y_{j_m})\,\bar{\xi}(x_{i_1})\cdots\bar{\xi}(x_{i_n})\xi(y_{j_1})\cdots\xi(y_{j_m}) \\ &\overset{\text{Here}}{\equiv} \mathrm{P}_0 + \mathrm{P}_2 + \int_{\mathrm{Q}_1,\ldots,\mathrm{Q}_4} \sum_{j=0}^{100} \gamma_j^\Lambda(\mathrm{Q}_1,\mathrm{Q}_2,\mathrm{Q}_3,\mathrm{Q}_4)\,\bar{\xi}_{j_1}(\mathrm{Q}_1)\bar{\xi}_{j_2}(\mathrm{Q}_2)\xi_{j_3}(\mathrm{Q}_3)\xi_{j_4}(\mathrm{Q}_4) + \mathrm{P}_{\geq 4}\end{aligned}$$

and focus on a quartic term

$$\int_{\mathrm{Q}_1,\ldots,\mathrm{Q}_4} \gamma_j^\Lambda(\mathrm{Q}_1,\mathrm{Q}_2,\mathrm{Q}_3,\mathrm{Q}_4)\,\bar{\xi}_{j_1}(\mathrm{Q}_1)\bar{\xi}_{j_2}(\mathrm{Q}_2)\xi_{j_3}(\mathrm{Q}_3)\xi_{j_4}(\mathrm{Q}_4).$$

The translation-invariance and our ansatz imply

$$\gamma_j^\Lambda(\mathrm{Q}_1,\mathrm{Q}_2,\mathrm{Q}_3,\mathrm{Q}_4) \overset{\text{Here}}{\equiv} \delta(\mathrm{Q}_1+\mathrm{Q}_2-(\mathrm{Q}_3+\mathrm{Q}_4))\,\gamma_j(\Lambda),$$

such that counting the powers

- $\mathrm{d}\,\mathrm{Q} \sim k^3$,
- $\delta(\mathrm{Q}_1+\mathrm{Q}_2-(\mathrm{Q}_3+\mathrm{Q}_4)) \sim \frac{1}{k^3}$,
- $\gamma_j(\Lambda) \sim k^r$,
- $\bar{\xi}_{j_i} \sim \xi_{j_i} \sim k^{-5/2}$,

gives a *relevant* coupling

$$0 = 4 \cdot 3 - 3 + r - 4 \cdot 5/2 \quad \Longleftrightarrow \quad r = 1.$$

Therefore we arrive at dimensionless couplings $\tilde{\gamma}$ by division of Λ:

$$\tilde{\gamma}_\bullet(\mathrm{Q}_1,\mathrm{Q}_2,\mathrm{Q}_3,\mathrm{Q}_4) = \frac{\gamma_\bullet(\mathrm{Q}_1,\mathrm{Q}_2,\mathrm{Q}_3,\mathrm{Q}_4)}{\Lambda} \quad \Longleftrightarrow \quad \gamma_\bullet(\mathrm{Q}_1,\mathrm{Q}_2,\mathrm{Q}_3,\mathrm{Q}_4) = \Lambda \cdot \tilde{\gamma}_\bullet(\mathrm{Q}_1,\mathrm{Q}_2,\mathrm{Q}_3,\mathrm{Q}_4) \tag{6.5}$$

implying

$$\frac{\mathrm{d}}{\mathrm{d}\Lambda}\gamma_\bullet = \Lambda \cdot \frac{\mathrm{d}}{\mathrm{d}\Lambda}\tilde{\gamma}_\bullet + \tilde{\gamma}_\bullet. \tag{6.6}$$

6.2.2 β-Functions and Fixed Point

Theorem 6.4 The flow equations in eq. (6.4) lead to the following β-functions for the dimensionless couplings

$$\begin{cases} \beta_{11}(\tilde{\gamma}_{11}, \tilde{\gamma}_{66}, \tilde{\gamma}_{91}, \tilde{\gamma}_{97}) & \approx & -\tilde{\gamma}_{11}/\Lambda \\ \beta_{66}(\tilde{\gamma}_{11}, \tilde{\gamma}_{66}, \tilde{\gamma}_{91}, \tilde{\gamma}_{97}) & \approx & -\tilde{\gamma}_{66}/\Lambda \\ \beta_{91}(\tilde{\gamma}_{11}, \tilde{\gamma}_{66}, \tilde{\gamma}_{91}, \tilde{\gamma}_{97}) & \approx & -\tilde{\gamma}_{91}/\Lambda + \tilde{\gamma}_{11} \cdot \left(4\, \widetilde{D}_{\Lambda,\epsilon}\right) \cdot \tilde{\gamma}_{66} \\ \beta_{97}(\tilde{\gamma}_{11}, \tilde{\gamma}_{66}, \tilde{\gamma}_{91}, \tilde{\gamma}_{97}) & \approx & -\tilde{\gamma}_{97}/\Lambda + \tilde{\gamma}_{97} \cdot \left(-16\, \widetilde{D}_{\Lambda,\epsilon}\right) \cdot \tilde{\gamma}_{66} \end{cases},$$

with $\widetilde{D}_{\Lambda,\epsilon} = \Lambda \cdot D_{\Lambda,\epsilon}$.

Proof 6.4 Inserting eqs. (6.5) and (6.6) into the flow-equations (6.4) yields the desired result.

Corollary 6.5 The vector $(\tilde{\gamma}^*_{11}, \tilde{\gamma}^*_{66}, \tilde{\gamma}^*_{91}, \tilde{\gamma}^*_{97}) \equiv (0, 0, 0, 0)$ is the only fixed point. Therefore (within our approximations) all the critical exponents equal to zero and we have found an instable fixed point.

Proof 6.5 Equating all β-functions to zero, and inserting the first two equations into the last two (with the order indicated above) gives the desired result.

7 Conclusion

This thesis was supposed to answer two questions:

1. *"Can one generalize the concept of universality to fermionic systems away from thermodynamic equilibrium?"*
2. *"If so, can we determine non-trivial critical exponents for the system in the literature [15] (performing a topological phase transition) when we combine a Mean Field Theory with a One-Loop-FRG?"*

The first question was answered in the affirmative:

Having found the translation of the partition function to the functional integral formalism yielded access to all the tools from quantum field theory. Generalizing the mathematical and physical concepts from the various literatures to arbitrary Graßmannian functional integrals that one can encounter (by this translation) was straightforward.

Also, the system that we studied from chapter 4 to chapter 6 featured the typical hallmark from FRG-analyses: Focusing on divergencies yielded an enormous reduction of complexity (infinitely many coupling constants that might be considered) to simplicity (in our case: four flow equations that needed to be solved).

The second question was answered in the negative:

The system of interest was introduced in formal correspondence to a topological insulator and yielded an action with quartic terms. After determining the self-energy in first order perturbation theory (the Born-Approximation from [17]), and demanding self-consistency of this approximation, we easily realized that the interaction terms corresponded to relevant couplings.

Finding divergent One-Loop-corrections (these are typical for systems performing phase transitions) to the effective interactions marked the starting point of the FRG-analysis. Building on the inherent analogy from FRG to Wilson's RG-scheme, we also conjectured, due to the positive scaling dimension of the couplings, the existence of a non-trivial ("Wilson-Fisher") fixed point for the effective interactions. What we found was an instable Gaussian fixed point (yielding trivial critical exponents).

Assuming the existence of a non-trivial fixed point for the effective interactions, the insight of our calculation is that *a One-Loop-FRG with our ansatz is not suited* to find it.

Elaborating the aforementioned analogy to Wilson's RG-scheme, we might consider us in a situation similar as with the $O(1)$-model in d dimensions, as in [5], after working through the first order pertubation theory. Now going over to the second order perturbation theory as in [5], might correspond in our case to including more vertex-functions to our ansatz in section 3.3.3.

Of course, the inclusion of vertex-functions corresponding to terms in the series expansion that are higher than quartic is unreasonable, as they scale to zero when the FRG is iterated. But it might be an option to conclude (from symmetry-considerations) which vertex-functions are theoretically allowed but not represented as One-Loop-divergencies. Especially those vertex-functions that differ by diverging couplings exclusively by the action of a symmetry-transformation might be interesting to include into our ansatz.

8 Appendix

8.1 Proofs

8.1.1 Proofs to Chapter 3

Proof 8.1 of Lemma 3.4 The definition of the trace reads

$$Z = \operatorname{Tr} \rho(t_{\mathrm{f}}) \equiv \operatorname{Tr} \rho(t_N) = \sum_I \langle I | \rho(t_N) | I \rangle.$$

Now, we include a resolution of the identity (cf. eq. (2.1)) and obtain[1]

$$\begin{aligned}
\sum_I \langle I | \rho(t_N)\, \mathbb{1}\, | I \rangle &= \int \mathrm{d}\psi\, \mathrm{e}^{-|\psi|^2} \sum_I \langle I | \rho(t_N) | \psi \rangle \langle \psi | I \rangle \\
&= \int \mathrm{d}\psi\, \mathrm{e}^{-|\psi|^2} \sum_I \langle \psi | I \rangle \langle I | \rho(t_N) | -\psi \rangle \\
&= \int \mathrm{d}\psi\, \mathrm{e}^{-|\psi|^2} \langle \psi | \underbrace{\sum_I |I\rangle\langle I|}_{=\mathbb{1}} \rho(t_N) | -\psi \rangle \\
&= \int \mathrm{d}\psi\, \mathrm{e}^{-|\psi|^2} \langle \psi | \rho(t_N) | -\psi \rangle \\
&\equiv \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-|\psi_{-,N}|^2} \langle \psi_{-,N} | \rho(t_N) | -\psi_{-,N} \rangle.
\end{aligned}$$

Inserting another resolution of identity

$$\mathbb{1}_{+,N} \equiv \int \mathrm{d}\psi_{+,N} \mathrm{e}^{-\overline{\psi}_{+,N}\psi_{+,N}} |\psi_{+,N}\rangle\langle\psi_{+,N}|$$

[1] Regarding the second equation sign: In order to "find" this minus sign in the derivation of the grand canonical partition function for fermionic many-body quantum systems as in [1], one has to use the block-diagonality of the second quantized Hamilton-operator w.r.t. the number of particles. Here one has to give a similar argument: From eq. (3.1) it is apparent that the action of the Liouville-operator $\mathscr{L}$ does not change the ratio of creation- to annihilation-operators in a monom of the type

$$c_{i_1}^\dagger \cdots c_{i_n}^\dagger |\text{ vac. }\rangle\langle\text{ vac. }| c_{j_1} \cdots c_{j_m},$$

although the action of the Liouville-operator on such a monom can create more than one term. Likewise the action of any power of the Liouville-operator (as in a series-expansion of eq. (3.2)) does not change this ratio. In the spirit of Hypothesis 3.7, we choose our initial state as in Remark 3.7.1. The argument (i. e. the derivation of the minus sign) is completed analogously to the aforementioned derivation in [1].

yields the desired result:

$$\begin{aligned} Z &= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}}\, \langle \psi_{-,N}|\, \mathbb{1}_{+,N}\, \rho(t_N)|-\psi_{-,N}\rangle \\ &\equiv \int \mathrm{d}\psi_{\pm,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}-\overline{\psi}_{+,N}\psi_{+,N}+\overline{\psi}_{-,N}\psi_{+,N}}\, \langle \psi_{+,N}|\rho(t_N)|-\psi_{-,N}\rangle \\ &= \int \mathrm{d}\psi_{\pm,N}\, \langle \psi_{+,N}|\rho_N|-\psi_{-,N}\rangle\, \mathrm{e}^{-\left(|\psi_{+,N}|^2+|\psi_{-,N}|^2-\overline{\psi}_{-,N}\psi_{+,N}\right)} \\ &= \int \mathrm{d}\psi_{\pm,N}\, \langle \psi_{+,N}|\rho_N|-\psi_{-,N}\rangle\, \mathrm{e}^{-S_N}. \end{aligned}$$

Proof 8.2 of Lemma 3.5

With eq. (3.2) we conclude together with the identity

$$\mathrm{e}^{\Delta t\, x} = 1 + \Delta t\, x + \mathcal{O}(\Delta t^2) \tag{8.1}$$

the equation

$$\langle \psi_{+,n+1}|\rho_{n+1}|-\psi_{-,n+1}\rangle = \langle \psi_{+,n+1}|\rho_n|-\psi_{-,n+1}\rangle + \Delta t\, \langle \psi_{+,n+1}|\mathscr{L}[\rho_n]|-\psi_{-,n+1}\rangle + \mathcal{O}(\Delta t^2). \tag{8.2}$$

Furthermore, one has the decomposition

$$\rho_n = \int \mathrm{d}\psi_{\pm,n}\, \mathrm{e}^{-\left(|\psi_{+,n}|^2+|\psi_{-,n}|^2\right)}\, |\psi_{+,n}\rangle\langle -\psi_{-,n}|\, \langle \psi_{+,n}|\rho_n|-\psi_{-,n}\rangle.$$

Then, it follows for the first term on the r.h.s. in eq. (8.2)

$$\begin{aligned} \langle \psi_{+,n+1}|\rho_n|-\psi_{-,n+1}\rangle &= \int \mathrm{d}\psi_{\pm,n}\, \mathrm{e}^{-\left(|\psi_{+,n}|^2+|\psi_{-,n}|^2\right)}\, \langle \psi_{+,n+1}|\psi_{+,n}\rangle\langle -\psi_{-,n}|-\psi_{-,n+1}\rangle\, \langle \psi_{+,n}|\rho_n|-\psi_{-,n}\rangle \\ &= \int \mathrm{d}\psi_{\pm,n}\, \mathrm{e}^{\Delta t\left[(\partial_t\overline{\psi}_{+,n})\psi_{+,n}+\overline{\psi}_{-,n}\partial_t\psi_{-,n}\right]}\, 1\, \langle \psi_{+,n}|\rho_n|-\psi_{-,n}\rangle, \end{aligned}$$

with discrete partial derivatives

$$\partial_t\psi_{-,n} \equiv \frac{\psi_{-,n+1}-\psi_{-,n}}{\Delta t}, \qquad \text{etc.}$$

For the second term on the r. h. s. in eq. (8.2) we obtain quite analogously

$$\begin{aligned} &\Delta t\, \langle \psi_{+,n+1}|\mathscr{L}[\rho_n]|-\psi_{-,n+1}\rangle \\ &= \int \mathrm{d}\psi_{\pm,n}\, \mathrm{e}^{-\left(\overline{\psi}_{+,n}\psi_{+,n}+\overline{\psi}_{-,n}\psi_{-,n}\right)}\, \Delta t\, \langle \psi_{+,n+1}|\mathscr{L}\left[|\psi_{+,n}\rangle\langle -\psi_{-,n}|\right]|-\psi_{-,n+1}\rangle\, \langle \psi_{+,n}|\rho_n|-\psi_{-,n}\rangle = \dots \end{aligned}$$

which can be further expanded with the definition of the Liouville-operator in eq. (3.1) to be

$$\begin{aligned}\ldots = \int \mathrm{d}\psi_{\pm,n}\, \mathrm{e}^{-(\overline{\psi}_{+,n}\psi_{+,n}+\overline{\psi}_{-,n}\psi_{-,n})}\Delta t \sum_\alpha \{ & 2\langle\psi_{+,n+1}|L_\alpha|\psi_{+,n}\rangle\langle-\psi_{-,n}|L_\alpha^\dagger|-\psi_{-,n+1}\rangle \\ & - \langle\psi_{+,n+1}|L_\alpha^\dagger L_\alpha|\psi_{+,n}\rangle\langle-\psi_{-,n}|-\psi_{-,n+1}\rangle \\ & - \langle\psi_{+,n+1}|\psi_{+,n}\rangle\langle-\psi_{-,n}|L_\alpha^\dagger L_\alpha|-\psi_{-,n+1}\rangle \\ & \}\langle\psi_{+,n}|\rho_n|-\psi_{-,n}\rangle \\ = \ldots\end{aligned}$$

or, with the abbreviations introduced above:

$$\ldots = \int \mathrm{d}\psi_{\pm,n}\, \mathrm{e}^{\Delta t\left[(\partial_t\overline{\psi}_{+,n})\psi_{+,n}+\overline{\psi}_{-,n}\partial_t\psi_{-,n}\right]}\Delta t\, \mathscr{L}(\overline{\psi}_{\pm,n+1},\psi_{\pm,n})\,\langle\psi_{+,n}|\rho_n|-\psi_{-,n}\rangle.$$

If we now use eq. (8.1) again, we get in total

$$\begin{aligned}\langle\psi_{+,n+1}|\rho_{n+1}|-\psi_{-,n+1}\rangle &= \langle\psi_{+,n+1}|\rho_n|-\psi_{-,n+1}\rangle + \Delta t\,\langle\psi_{+,n+1}|\mathscr{L}[\rho_n]|-\psi_{-,n+1}\rangle + \mathcal{O}(\Delta t^2) \\ &= \int \mathrm{d}\psi_{\pm,n}\mathrm{e}^{\Delta t\left[(\partial_t\overline{\psi}_{+,n})\psi_{+,n}+\overline{\psi}_{-,n}\partial_t\psi_{-,n}\right]}\mathrm{e}^{\Delta t\,\mathscr{L}(\overline{\psi}_{\pm,n+1},\psi_{\pm,n})}\langle\psi_{+,n}|\rho_n|-\psi_{-,n}\rangle + \mathcal{O}(\Delta t^2) \\ &= \int \mathrm{d}\psi_{\pm,n}\mathrm{e}^{\Delta t\left[(\partial_t\overline{\psi}_{+,n})\psi_{+,n}+\overline{\psi}_{-,n}\partial_t\psi_{-,n}+\mathscr{L}(\overline{\psi}_{\pm,n+1},\psi_{\pm,n})\right]}\langle\psi_{+,n}|\rho_n|-\psi_{-,n}\rangle + \mathcal{O}(\Delta t^2) \\ &= \int \mathrm{d}\psi_{\pm,n}\mathrm{e}^{-\Delta t\,S_{[n,n+1]}}\langle\psi_{+,n}|\rho_n|-\psi_{-,n}\rangle + \mathcal{O}(\Delta t^2) \\ &= \int \mathrm{d}\psi_{\pm,n}\,\langle\psi_{+,n}|\rho_n|-\psi_{-,n}\rangle\,\mathrm{e}^{-\Delta t\,S_{[n,n+1]}} + \mathcal{O}(\Delta t^2),\end{aligned}$$

the result, which was needed to be proven.

Proof 8.3 of Corollary 3.6 This follows immediately by looking at the concrete form of the Liouville-superoperator $\mathscr{L}(\overline{\psi}_{\pm,n+1},\psi_{\pm,n})$ in the proof of Lemma 3.5. Neglecting the infinitesimal time-differences gives exactly this result.

Proof 8.4 of Lemma 3.8

Preparation: For any operators $A,\ B$ one has

$$\begin{aligned}\langle AB\rangle &= \mathrm{Tr}(AB\,\rho_N) \\ &= \sum_n \langle n|AB\,\rho_N|n\rangle \\ &= \sum_n \langle n|\mathbb{1}_{-,N}AB\,\rho_N|n\rangle \\ &= \sum_n \int \mathrm{d}\psi_{-,N}\,\mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}}\langle n|\psi_{-,N}\rangle\langle\psi_{-,N}|AB\,\rho_N|n\rangle \\ &= \int \mathrm{d}\psi_{-,N}\,\mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}}\langle\psi_{-,N}|AB\,\rho_N\sum_n|n\rangle\langle n|-\psi_{-,N}\rangle \\ &= \int \mathrm{d}\psi_{-,N}\,\mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}}\langle\psi_{-,N}|AB\,\rho_N|-\psi_{-,N}\rangle.\end{aligned}$$

With the cyclicity of the trace one concludes

1. $\langle AB\rangle = \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}|\, AB\, \rho_N\, |-\psi_{-,N}\rangle$,
2. $\langle AB\rangle = \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}|\, B\, \rho_N\, A\, |-\psi_{-,N}\rangle$,
3. $\langle AB\rangle = \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}|\, \rho_N\, AB\, |-\psi_{-,N}\rangle$.

The identity $\langle \psi_{+,N}\overline{\psi}_{-,N}\rangle = -\langle a^\dagger a\rangle$.

We use the first equation with $A = a^\dagger$, $B = a$ and insert a resolution of identity between density and annihilation-operator:

$$\begin{aligned}
\langle a^\dagger a\rangle &= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}|a^\dagger a\rho_N|-\psi_{-,N}\rangle \\
&= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}|a\rho_N|-\psi_{-,N}\rangle \left(-\overline{\psi}_{-,N}\right) \\
&= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}|a\mathbb{1}_{+,N}\rho_N|-\psi_{-,N}\rangle \left(-\overline{\psi}_{-,N}\right) \\
&= \int \mathrm{d}\psi_{\pm,N}\, \langle \psi_{+,N}|\rho_N|-\psi_{-,N}\rangle\, \mathrm{e}^{-S_N} \left(-\psi_{+,N}\overline{\psi}_{-,N}\right) \\
&= \dots \\
&= -\langle \psi_{+,N}\overline{\psi}_{-,N}\rangle,
\end{aligned}$$

where in the last step the calculation was continued analogously to the proof of Theorem 3.3.

The identity: $\langle \psi_{-,N}\overline{\psi}_{+,N}\rangle = 1 - \langle a^\dagger a\rangle$.

We use the second equation with $B = a^\dagger$ and $A = a$ and insert another resolution of identity:

$$\begin{aligned}
\langle aa^\dagger\rangle &= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}|\, a^\dagger\, \rho_N\, a\, |-\psi_{-,N}\rangle \\
&= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}|\mathbb{1}_{+,N}a^\dagger\, \rho_N\, |-\psi_{-,N}\rangle \left(-\psi_{-,N}\right) \\
&= \int \mathrm{d}\psi_{\pm,N}\, \langle \psi_{+,N}|\rho_N|-\psi_{-,N}\rangle\, \mathrm{e}^{-S_N} \left(+\psi_{-,N}\overline{\psi}_{+,N}\right) \\
&= \dots \\
&= \langle \psi_{-,N}\overline{\psi}_{+,N}\rangle,
\end{aligned}$$

where in the last step the calculation was continued analogously to the proof of Theorem 3.3.

The identity: $\langle \psi_{+,N-1}\overline{\psi}_{+,N}\rangle = -\langle a^\dagger a\rangle$.

We use the first equation with $A = a^\dagger$ and $B = a$:

$$\begin{aligned}
\langle a^\dagger a\rangle &= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}| a^\dagger a \rho_N| -\psi_{-,N}\rangle \\
&= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}| \mathbb{1}_{+,N} a^\dagger a \rho_N| -\psi_{-,N}\rangle \\
&= \int \mathrm{d}\psi_{\pm,N}\, \langle \psi_{+,N}|\, a\, \rho_N| -\psi_{-,N}\rangle\, \mathrm{e}^{-S_N} \left(-\overline{\psi}_{+,N}\right) \\
&= \int \mathrm{d}\psi_{\pm,N}\, a\, \langle \psi_{+,N}|\rho_N| -\psi_{-,N}\rangle\, \mathrm{e}^{-S_N} \left(-\overline{\psi}_{+,N}\right) \\
&\overset{\text{Lemma 3.5}}{=} \int \mathrm{d}\psi_{\pm,N}\, \mathrm{d}\psi_{\pm,N-1}\, a\, \langle \psi_{+,N-1}|\rho_{N-1}| -\psi_{-,N-1}\rangle\, \mathrm{e}^{-\Delta t\, S_{[N-1,N]}}\, \mathrm{e}^{-S_N} \left(-\overline{\psi}_{+,N}\right) + \mathcal{O}(\Delta t^2) \\
&= \int \mathrm{d}\psi_{\pm,N}\, \mathrm{d}\psi_{\pm,N-1}\, \langle \psi_{+,N-1}|\rho_{N-1}| -\psi_{-,N-1}\rangle\, \mathrm{e}^{-\Delta t\, S_{[N-1,N]}}\, \mathrm{e}^{-S_N} \left(-\psi_{+,N-1}\overline{\psi}_{+,N}\right) + \mathcal{O}(\Delta t^2) \\
&= \dots \\
&= -\langle \psi_{+,N-1}\overline{\psi}_{+,N}\rangle,
\end{aligned}$$

where the identities

$$a\, \mathrm{e}^{+\overline{\psi}_{+,N}\cdot\psi_{+,N-1}} = a\, \langle\, \psi_{+,N}\,|\,\psi_{+,N-1}\,\rangle = \langle\, \psi_{+,N}\,|\, a\,|\,\psi_{+,N-1}\,\rangle = \langle\, \psi_{+,N}\,|\,\psi_{+,N-1}\,\rangle\, \psi_{+,N-1} = \mathrm{e}^{+\overline{\psi}_{+,N}\cdot\psi_{+,N-1}}\, \psi_{+,N-1}$$

have been used and in the last step the calculation was continued analogously to the proof of Theorem 3.3.

The identity: $\langle \psi_{+,N}\overline{\psi}_{+,N}\rangle = 1 - \langle a^\dagger a\rangle$.

We use the first equation with $A = a$ and $B = a^\dagger$:

$$\begin{aligned}
\langle a a^\dagger\rangle &= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}|\, a\, a^\dagger\, \rho_N\, | -\psi_{-,N}\rangle \\
&= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N}|\, a \mathbb{1}_{+,N} a^\dagger\, \rho_N\, | -\psi_{-,N}\rangle \\
&= \int \mathrm{d}\psi_{\pm,N}\, \langle \psi_{+,N}|\rho_N| -\psi_{-,N}\rangle\, \mathrm{e}^{-S_N} \left(\psi_{+,N}\overline{\psi}_{+,N}\right) \\
&= \dots \\
&= \langle \psi_{+,N}\overline{\psi}_{+,N}\rangle,
\end{aligned}$$

where in the last step the calculation was continued analogously to the proof of Theorem 3.3.

The identity: $\langle \psi_{-,N} \overline{\psi}_{-,N} \rangle = \langle a a^\dagger \rangle$.

We use the second equation with $A = a$, $B = a^\dagger$. Then it follows

$$\begin{aligned}
\langle a a^\dagger \rangle &= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N} |\, a^\dagger\, \rho_N\, a \,| -\psi_{-,N} \rangle \\
&= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N} |\, \rho_N \,| -\psi_{-,N} \rangle \left(\psi_{-,N} \overline{\psi}_{-,N} \right) \\
&= \int \mathrm{d}\psi_{\pm,N}\, \langle \psi_{+,N} |\, \rho_N |\, -\psi_{-,N} \rangle\, \mathrm{e}^{-S_N} \left(\psi_{-,N} \overline{\psi}_{-,N} \right) \\
&= \dots \\
&= \langle \psi_{-,N} \overline{\psi}_{-,N} \rangle,
\end{aligned}$$

where in the last step the calculation was continued analogously to the proof of Theorem 3.3.

The identity: $\langle \psi_{-,N} \overline{\psi}_{-,N-1} \rangle = -\langle a^\dagger a \rangle$.

We use the third equation with $A = a^\dagger$ und $B = a$:

$$\begin{aligned}
\langle a^\dagger a \rangle &= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N} | \rho_N a^\dagger a | -\psi_{-,N} \rangle \\
&= \int \mathrm{d}\psi_{-,N}\, \mathrm{e}^{-\overline{\psi}_{-,N}\psi_{-,N}} \langle \psi_{-,N} | \mathbb{1}_{+,N} \rho_N | -\psi_{-,N} \rangle \cdot \left(-a^\dagger \psi_{-,N} \right) \\
&= \int \mathrm{d}\psi_{\pm,N}\, \langle \psi_{+,N} | \rho_N | -\psi_{-,N} \rangle\, \mathrm{e}^{-S_N} \cdot \left(-a^\dagger \psi_{-,N} \right) \\
&\overset{\text{Lemma 3.5}}{=} \int \mathrm{d}\psi_{\pm,N}\, \mathrm{d}\psi_{\pm,N-1} \langle \psi_{+,N-1} | \rho_{N-1} | -\psi_{-,N-1} \rangle\, \mathrm{e}^{-\Delta t\, S_{[N-1,N]}}\, \mathrm{e}^{-S_N} \cdot \left(-a^\dagger \psi_{-,N} \right) + \mathcal{O}(\Delta t^2) \\
&= \int \mathrm{d}\psi_{\pm,N}\, \mathrm{d}\psi_{\pm,N-1}\, \langle \psi_{+,N-1} | \rho_{N-1} | -\psi_{-,N-1} \rangle\, \mathrm{e}^{-\Delta t\, S_{[N-1,N]}}\, \mathrm{e}^{-S_N} \cdot \left(\overline{\psi}_{-,N-1} \psi_{-,N} \right) + \mathcal{O}(\Delta t^2) \\
&= \int \mathrm{d}\psi_{\pm,N}\, \mathrm{d}\psi_{\pm,N-1}\, \langle \psi_{+,N-1} | \rho_{N-1} | -\psi_{-,N-1} \rangle\, \mathrm{e}^{-\Delta t\, S_{[N-1,N]}}\, \mathrm{e}^{-S_N} \cdot \left(-\psi_{-,N} \overline{\psi}_{-,N-1} \right) + \mathcal{O}(\Delta t^2) \\
&= \dots \\
&= -\langle \psi_{-,N} \overline{\psi}_{-,N-1} \rangle,
\end{aligned}$$

where the identities

$$\begin{aligned}
\mathrm{e}^{+\overline{\psi}_{-,N-1} \cdot \psi_{-,N}}\, a^\dagger &= \langle\, -\psi_{-,N-1} \,|\, -\psi_{-,N} \,\rangle\, a^\dagger \\
&= \langle\, -\psi_{-,N-1} \,|\, a^\dagger \,|\, -\psi_{-,N} \,\rangle \\
&= \langle\, -\psi_{-,N-1} \,|\, -\psi_{-,N} \,\rangle \cdot \left(-\overline{\psi}_{-,N-1} \right) \\
&= \mathrm{e}^{+\overline{\psi}_{-,N-1} \cdot \psi_{-,N}} \left(-\overline{\psi}_{-,N-1} \right)
\end{aligned}$$

have been used and in the last step the calculation was continued analogously to the proof of Theorem 3.3.

Proof 8.5 of Remark 3.11.1. One has

$$\begin{aligned}\frac{\delta}{\delta\phi(\bullet)}\Gamma[\phi,\overline{\phi}] &= \sum\!\!\!\!\!\!\int \left(\frac{\delta}{\delta j(x)}W[j,\bar{j}],\frac{\delta}{\delta\bar{j}(x)}W[j,\bar{j}]\right)\cdot\begin{pmatrix}\frac{\delta j(x)}{\delta\phi(\bullet)}\\ \frac{\delta\bar{j}(x)}{\delta\phi(\bullet)}\end{pmatrix}+\bar{j}(\bullet)+\sum\!\!\!\!\!\!\int\left(\frac{\delta\bar{j}(x)}{\delta\phi(\bullet)}\phi(x)+\overline{\phi}(x)\frac{\delta j(x)}{\delta\phi(\bullet)}\right)\\ &= \sum\!\!\!\!\!\!\int\left(-\overline{\phi}(x),\phi(x)\right)\cdot\begin{pmatrix}\frac{\delta j(x)}{\delta\phi(\bullet)}\\ \frac{\delta\bar{j}(x)}{\delta\phi(\bullet)}\end{pmatrix}+\bar{j}(\bullet)+\sum\!\!\!\!\!\!\int\left(\frac{\delta\bar{j}(x)}{\delta\phi(\bullet)}\phi(x)+\overline{\phi}(x)\frac{\delta j(x)}{\delta\phi(\bullet)}\right)\\ &= \bar{j}(\bullet),\end{aligned}$$

and

$$\begin{aligned}\frac{\delta}{\delta\overline{\phi}(\bullet)}\Gamma[\phi,\overline{\phi}] &= \sum\!\!\!\!\!\!\int \left(\frac{\delta}{\delta j(x)}W[j,\bar{j}],\frac{\delta}{\delta\bar{j}(x)}W[j,\bar{j}]\right)\cdot\begin{pmatrix}\frac{\delta j(x)}{\delta\overline{\phi}(\bullet)}\\ \frac{\delta\bar{j}(x)}{\delta\overline{\phi}(\bullet)}\end{pmatrix}-j(\bullet)+\sum\!\!\!\!\!\!\int\left(\frac{\delta\bar{j}(x)}{\delta\overline{\phi}(\bullet)}\phi(x)+\overline{\phi}(x)\frac{\delta j(x)}{\delta\overline{\phi}(\bullet)}\right)\\ &= -j(\bullet).\end{aligned}$$

Proof 8.6 of Lemma 3.14.

For the first equation we compute

$$\begin{aligned}W[j,\bar{j}] &= -\log\int\mathscr{D}(\psi,\overline{\psi})\exp\left(-S[\psi,\overline{\psi}]+\sum\!\!\!\!\!\!\int(\bar{j}\psi+\overline{\psi}j)\right)\\ &= -\log\int\mathscr{D}(M\cdot\psi,\overline{M}\cdot\overline{\psi})\exp\left(-S[M\cdot\psi,\overline{M}\cdot\overline{\psi}]+\sum\!\!\!\!\!\!\int_x\bar{j}(x)\,(M\cdot\psi)(x)+(\overline{M}\cdot\overline{\psi})(x)\,j(x)\right)\\ &= \dots\end{aligned}$$

using now

$$\sum\!\!\!\!\!\!\int_x\bar{j}(x)\,(M\cdot\psi)(x)=\sum\!\!\!\!\!\!\int_x\sum\!\!\!\!\!\!\int_y\bar{j}(x)\,M(x,y)\,\psi(y)=\sum\!\!\!\!\!\!\int_y\sum\!\!\!\!\!\!\int_x M^{\mathrm{T}}(y,x)\bar{j}(x)\,\psi(y)=\sum\!\!\!\!\!\!\int_y(M^{\mathrm{T}}\cdot\bar{j})(y)\,\psi(y)$$

and analogously one has

$$\sum\!\!\!\!\!\!\int_x(\overline{M}\cdot\overline{\psi})(x)\,j(x)=\sum\!\!\!\!\!\!\int_y\overline{\psi}(y)\,(\overline{M}^{\mathrm{T}}\cdot j)(y).$$

Then, it follows (together with Definition 3.13)

$$\begin{aligned}\dots &= -\log\int\mathscr{D}(\psi,\overline{\psi})\exp\left(-S[\psi,\overline{\psi}]+\sum\!\!\!\!\!\!\int_y(M^{\mathrm{T}}\cdot\bar{j})(y)\,\psi(y)+\overline{\psi}(y)\,(\overline{M}^{\mathrm{T}}\cdot j)(y)\right)\\ &= W[\overline{M}^{\mathrm{T}}\cdot j,M^{\mathrm{T}}\cdot\bar{j}].\end{aligned}$$

For the second equation one has

$$\begin{aligned}\phi[\bullet, \overline{M}^{\mathrm{T}} \cdot j, M^{\mathrm{T}} \cdot \bar{j}] &= -\frac{\vec{\delta}}{\delta(M^{\mathrm{T}} \cdot \bar{j})(\bullet)} W[\overline{M}^{\mathrm{T}} \cdot j, M^{\mathrm{T}} \cdot \bar{j}] \\ &= -\frac{\vec{\delta}}{\delta(M^{\mathrm{T}} \cdot \bar{j})(\bullet)} W[j, \bar{j}] \\ &= \dots\end{aligned}$$

Now one calculates

$$\bar{j}(x) = (M^{-\mathrm{T}} \cdot M^{\mathrm{T}} \cdot \bar{j})(x) = \sum\!\!\!\!\!\!\int_y M^{-\mathrm{T}}(x, y)\, (M^{\mathrm{T}} \cdot \bar{j})(y)$$

and therefore

$$\frac{\vec{\delta}}{\delta(M^{\mathrm{T}} \cdot \bar{j})(\bullet)} = \sum\!\!\!\!\!\!\int_x \left(\frac{\vec{\delta}}{\delta(M^{\mathrm{T}} \cdot \bar{j})(\bullet)} \bar{j}(x) \right) \frac{\vec{\delta}}{\delta \bar{j}(x)} = \sum\!\!\!\!\!\!\int_x M^{-\mathrm{T}}(x, \bullet) \frac{\vec{\delta}}{\delta \bar{j}(x)} = \sum\!\!\!\!\!\!\int_x M^{-1}(\bullet, x) \frac{\vec{\delta}}{\delta \bar{j}(x)}.$$

It then follows with Notation 3.10

$$\dots = -\sum\!\!\!\!\!\!\int_x M^{-1}(\bullet, x) \left(\frac{\vec{\delta}}{\delta \bar{j}(x)} W[j, \bar{j}] \right) = +\sum\!\!\!\!\!\!\int_x M^{-1}(\bullet, x)\, \phi[x, j, \bar{j}] = \left(M^{-1} \cdot \phi \right)[\bullet, j, \bar{j}],$$

Analogously, one obtains

$$\overline{\phi}[\bullet, \overline{M}^{\mathrm{T}} \cdot j, M^{\mathrm{T}} \cdot \bar{j}] = \frac{\vec{\delta}}{\delta(\overline{M}^{\mathrm{T}} \cdot j)(\bullet)} W[j, \bar{j}] = \sum\!\!\!\!\!\!\int_x \overline{M}^{-1}(\bullet, x) \left(\frac{\vec{\delta}}{\delta j(x)} W[j, \bar{j}] \right) = \left(\overline{M}^{-1} \cdot \overline{\phi} \right)[\bullet, j, \bar{j}].$$

Proof 8.7 of Theorem 3.15.

With Lemma 3.14 one can prove the theorem very easily:

$$\begin{aligned}\Gamma[M^{-1} \cdot \phi, \overline{M}^{-1} \cdot \overline{\phi}] &= W[\overline{M}^{\mathrm{T}} \cdot j, M^{\mathrm{T}} \cdot \bar{j}] + \sum\!\!\!\!\!\!\int_x (M^{\mathrm{T}} \cdot \bar{j})(x)\, (M^{-1} \cdot \phi)(x) + (\overline{M}^{-1} \cdot \overline{\phi})(x)\, (\overline{M}^{\mathrm{T}} \cdot j)(x) \\ &= W[j, \bar{j}] + \sum\!\!\!\!\!\!\int_x \bar{j}(x)\, \phi(x) + \overline{\phi}(x)\, j(x) \\ &= \Gamma[\phi, \overline{\phi}].\end{aligned}$$

Proof 8.8 of Approximation 3.16.

Starting with eq. (3.6):

$$\begin{aligned}\Gamma[\phi,\overline{\phi}] &= W[j,\overline{j}] + \int\!\!\!\!\!\!\sum \overline{j}\phi + \overline{\phi}j \\ &= -\log Z[j,\overline{j}] + \int\!\!\!\!\!\!\sum \overline{j}\phi + \overline{\phi}j,\end{aligned}$$

We do the following things:

1. We multiply this equation by -1 and apply the exponential-function to both sides,
2. we insert the definition of the source-dependent partition function $Z[j,\overline{j}]$ from eq. (3.4),
3. we use the invariance of the functional integral by Graßmann-valued translations $\psi - \phi \to \Delta$, etc.

This gives in equations

$$\begin{aligned}\exp\left(-\Gamma[\phi,\overline{\phi}]\right) &= Z[j,\overline{j}]\exp\left(-\int\!\!\!\!\!\!\sum \overline{j}\phi + \overline{\phi}j\right) \\ &= \int \mathscr{D}(\psi,\overline{\psi})\exp\left(-S[\psi,\overline{\psi}] + \int\!\!\!\!\!\!\sum \overline{j}(\psi-\phi) + (\overline{\psi}-\overline{\phi})j\right) \\ &= \int \mathscr{D}(\Delta,\overline{\Delta})\exp\left(-S[\phi+\Delta,\overline{\phi}+\overline{\Delta}] + \int\!\!\!\!\!\!\sum \overline{j}\Delta + \overline{\Delta}j\right).\end{aligned}$$

The action S can now be expanded analogously to Notation 3.9 (again, in schematic notation) in the variation

$$\begin{aligned}S[\phi+\Delta,\overline{\phi}+\overline{\Delta}] = \quad & S[\phi,\overline{\phi}] \\ & + \int\!\!\!\!\!\!\sum \left(\frac{\delta}{\delta\phi}S[\phi,\overline{\phi}], \frac{\delta}{\delta\overline{\phi}}S[\phi,\overline{\phi}]\right)\cdot\begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix} \\ & + \frac{1}{2!}\int\!\!\!\!\!\!\sum\int\!\!\!\!\!\!\sum \begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix}^{\mathrm{T}}\cdot\begin{pmatrix}\frac{\delta^2}{\delta\phi\,\delta\phi}S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}S[\phi,\overline{\phi}] \\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}S[\phi,\overline{\phi}]\end{pmatrix}\cdot\begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix} \\ & + \mathcal{O}\left(\Delta^3, \Delta^2\overline{\Delta}^1, \Delta^1\overline{\Delta}^2, \overline{\Delta}^3\right).\end{aligned}$$

With this we get

$$\exp\left(-\Gamma[\phi,\overline{\phi}]\right) = \exp\left(-S[\phi,\overline{\phi}]\right)\int \mathscr{D}(\Delta,\overline{\Delta})\ \exp \mathrm{P}(\Delta,\overline{\Delta},\phi,\overline{\phi}) \tag{8.3}$$

with a polynomial P in the integration-variables and in the classical fields. In the so-called "Tree-Level-approximation" one now approximates[2]

$$\int \mathscr{D}(\Delta,\overline{\Delta})\ \exp \mathrm{P}(\Delta,\overline{\Delta},\phi,\overline{\phi}) \quad \approx \quad \mathcal{O}(1).$$

So, one assumes that the functional integral (which, when evaluated, is again a polynomial expression in the classical fields) gives essentially ϕ-independent contributions.

[2]The meaning of the symbol "$\approx$" is discussed in section 3.4.3.

After applying the logarithm and using the inverse property (also for Graßmann-valued polynomial expressions?!) we thus obtain

$$\Gamma[\phi,\overline{\phi}] \quad \approx \quad S[\phi,\overline{\phi}] + \text{const.}$$

Remembering now that the physically relevant part of the effective action is in the numerical values of the non-trivial correlators (and thus in the coefficients of the non-constant monomials in Γ (cf. the series-expansion in eq. (3.7) and section 3.4.1), then, one usually simply notes for convenience:

$$\Gamma[\phi,\overline{\phi}] \quad \approx \quad S[\phi,\overline{\phi}].$$

Proof 8.9 of Approximation 3.18.

The "One-Loop-approximation" builds on the Tree-Level-approximation. If we use Corollary 3.17, then (in the context of the Tree-Level-approximation) in eq. (8.3) the terms linear in Δ and $\overline{\Delta}$ cancel out. Then we can use this to connect to eq. (8.3), by neglecting the contributions to the polynomial P of order $\Delta^n\,\overline{\Delta}^m$ with $n+m \geq 3$:

$$\begin{aligned}\exp\left(-\Gamma[\phi,\overline{\phi}]\right) &\approx \exp\left(-S[\phi,\overline{\phi}]\right) \int \mathscr{D}(\Delta,\overline{\Delta})\, \exp\, -\frac{1}{2}\iint\iint \begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} \frac{\delta^2}{\delta\phi\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \end{pmatrix} \cdot \begin{pmatrix}\Delta\\ \overline{\Delta}\end{pmatrix} \\ &= \exp\left(-S[\phi,\overline{\phi}]\right) \mathrm{Pf} \begin{pmatrix} \frac{\delta^2}{\delta\phi\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \end{pmatrix}.\end{aligned}$$

If we now take the logarithm on both sides and use the inversion-property between exponential-function and logarithm (?!) and fundamental identities for the matrix logarithm (also for Graßmann-valued matrices?!), we get

$$\begin{aligned} -\Gamma[\phi,\overline{\phi}] &\approx -S[\phi,\overline{\phi}] + \log \mathrm{Pf} \begin{pmatrix} \frac{\delta^2}{\delta\phi\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \end{pmatrix} \\ &\overset{?!}{=} -S[\phi,\overline{\phi}] + \frac{1}{2}\log \mathrm{Det} \begin{pmatrix} \frac{\delta^2}{\delta\phi\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \end{pmatrix} \\ &\overset{?!}{=} -S[\phi,\overline{\phi}] + \frac{1}{2}\mathrm{Tr}\log \begin{pmatrix} \frac{\delta^2}{\delta\phi\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \end{pmatrix} \\ \Longleftrightarrow +\Gamma[\phi,\overline{\phi}] &\approx +S[\phi,\overline{\phi}] - \frac{1}{2}\mathrm{Tr}\log \begin{pmatrix} \frac{\delta^2}{\delta\phi\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S[\phi,\overline{\phi}] \end{pmatrix}. \end{aligned}$$

Proof 8.10 of Lemma 3.20.

First Identity

One has not only for the quadratic sector

$$\frac{\vec{\delta}^2}{\delta\overline{\psi}(y_1)\delta\psi(y_2)}\overline{\psi}(x_1)\,\psi(x_2) = -\delta(x_2-y_2)\delta(x_1-y_1) = -\frac{\delta^2}{\delta\overline{\psi}(y_1)\delta\psi(y_2)}\overline{\psi}(x_1)\,\psi(x_2),$$

but also for the quartic sector

$$\begin{aligned}-\frac{\vec{\delta}^2}{\delta\overline{\psi}(y_1)\delta\psi(y_2)}\overline{\psi}(x_1)\,\overline{\psi}(x_2)\,\psi(x_3)\,\psi(x_4) &= (\delta(x_1-y_1)\overline{\psi}(x_2)-\delta(x_2-y_1)\overline{\psi}(x_1))(\delta(x_4-y_2)\psi(x_3)-\delta(x_3-y_2)\psi(x_4))\\ &= \frac{\delta^2}{\delta\overline{\psi}(y_1)\delta\psi(y_2)}\overline{\psi}(x_1)\,\overline{\psi}(x_2)\,\psi(x_3)\,\psi(x_4),\end{aligned}$$

where the Delta-symbols denote a multiplication of a Kronecker-Delta and Dirac-Delta for the discrete and continuous part of the generalized indices.

Second Identity

Here, one also has for the quadratic sector

$$\frac{\vec{\delta}^2}{\delta\overline{\psi}(y_1)\delta\psi(y_2)}\overline{\psi}(x_1)\,\psi(x_2) = -\delta(x_2-y_2)\delta(x_1-y_1) = \frac{\delta^2}{\delta\psi(y_2)\delta\overline{\psi}(y_1)}\overline{\psi}(x_1)\,\psi(x_2),$$

but also in the quartic sector

$$\begin{aligned}\frac{\vec{\delta}^2}{\delta\overline{\psi}(y_1)\delta\psi(y_2)}\overline{\psi}(x_1)\,\overline{\psi}(x_2)\,\psi(x_3)\,\psi(x_4) &= (\delta(x_1-y_1)\overline{\psi}(x_2)-\delta(x_2-y_1)\overline{\psi}(x_1))(\delta(x_3-y_2)\psi(x_4)-\delta(x_4-y_2)\psi(x_3))\\ &= \frac{\delta^2}{\delta\psi(y_2)\delta\overline{\psi}(y_1)}\overline{\psi}(x_1)\,\overline{\psi}(x_2)\,\psi(x_3)\,\psi(x_4).\end{aligned}$$

The proof is finished by using the U(1) invariance of the action, implying[3] that in a series expansion of the fields each monomial possesses equally many ψ-variables as $\overline{\psi}$-variables. Then one considers a monomial of this type

$$\overline{\psi}(x_1)\cdots\overline{\psi}(x_n)\,\psi(\tilde{x}_1)\cdots\psi(\tilde{x}_n)$$

and proves the identities by induction[4] over the number $n \in \mathbb{N}_{>1}$. Noting the linearity of the derivates completes the proof.

Proof 8.11 of Approximation 3.21.

One separates the second variation according to

$$\begin{aligned}\begin{pmatrix}\frac{\delta^2}{\delta\phi\,\delta\phi}S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}S[\phi,\overline{\phi}]\\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}S[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}S[\phi,\overline{\phi}]\end{pmatrix} &= \begin{pmatrix}0 & -G_0^{-\mathrm{T}}\\ G_0^{-1} & 0\end{pmatrix} + \begin{pmatrix}\frac{\delta^2}{\delta\phi\,\delta\phi}S_{\mathrm{int}}[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}S_{\mathrm{int}}[\phi,\overline{\phi}]\\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}S_{\mathrm{int}}[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}S_{\mathrm{int}}[\phi,\overline{\phi}]\end{pmatrix}\\ &\equiv \qquad \mathcal{G}_0^{-1} \qquad + \qquad \mathcal{V}[\phi,\overline{\phi}].\end{aligned}$$

[3] See, for example, [7].

[4] As this is a little bit lengthy (but very easy), it is not documented in this thesis.

Using now that

$$\mathcal{G}_0 = \begin{pmatrix} 0 & -G_0^{-\mathrm{T}} \\ G_0^{-1} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & G_0 \\ -G_0^{\mathrm{T}} & 0 \end{pmatrix},$$

it follows, after simplifying the notation $\mathcal{V}[\phi, \overline{\phi}] \equiv \mathcal{V}$, that

$$\begin{aligned} \mathrm{Tr}\log\left[\mathcal{G}_0^{-1} + \mathcal{V}\right] &= \mathrm{Tr}\log\left[\mathcal{G}_0^{-1} \cdot \left(1 + \mathcal{G}_0\mathcal{V}\right)\right] \\ &\overset{?!}{=} \mathrm{Tr}\log\left[\mathcal{G}_0^{-1}\right] + \mathrm{Tr}\log\left[1 + \mathcal{G}_0 \cdot \mathcal{V}\right] \\ &= \mathcal{O}(1) + \mathrm{Tr}\log\left[1 + \mathcal{G}_0 \cdot \mathcal{V}\right]. \end{aligned}$$

The first term again provides a (in ϕ and $\overline{\phi}$) constant (and thus physically irrelevant) contribution. The second term provides (physically relevant) corrections. Therefore, we focus on the second term and expand the logarithm in its series representation for "small" corrections to the identity:

$$\begin{aligned} \mathrm{Tr}\log\left[1 + \mathcal{G}_0 \cdot \mathcal{V}\right] &\approx \mathrm{Tr}\left[\mathcal{G}_0 \cdot \mathcal{V} - \frac{(\mathcal{G}_0 \cdot \mathcal{V})^2}{2} \pm \dots\right] \\ &= \mathrm{Tr}\left[\mathcal{G}_0 \cdot \mathcal{V}\right] - \frac{1}{2}\mathrm{Tr}\left[\mathcal{G}_0 \cdot \mathcal{V} \cdot \mathcal{G}_0 \cdot \mathcal{V}\right] \pm \dots \end{aligned}$$

First summand. One computes

$$\begin{aligned} \mathcal{G}_0\mathcal{V} &= \begin{pmatrix} 0 & G_0 \\ -G_0^{\mathrm{T}} & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{\delta^2}{\delta\phi\,\delta\phi} S_{\mathrm{int}}[\phi, \overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S_{\mathrm{int}}[\phi, \overline{\phi}] \\ \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S_{\mathrm{int}}[\phi, \overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S_{\mathrm{int}}[\phi, \overline{\phi}] \end{pmatrix} \\ &= \begin{pmatrix} G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S_{\mathrm{int}}[\phi, \overline{\phi}] & G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S_{\mathrm{int}}[\phi, \overline{\phi}] \\ -G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\phi} S_{\mathrm{int}}[\phi, \overline{\phi}] & -G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S_{\mathrm{int}}[\phi, \overline{\phi}] \end{pmatrix}, \end{aligned}$$

where the formal products are to be evaluated by insertion of corresponding "unities".

Because of Lemma 3.20 one easily concludes that

$$\frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S_{\mathrm{int}}[\phi, \overline{\phi}] = -\left(\frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S_{\mathrm{int}}[\phi, \overline{\phi}]\right)^{\mathrm{T}}, \qquad \text{and} \qquad B^{\mathrm{T}}A^{\mathrm{T}} = (AB)^{\mathrm{T}}. \tag{8.4}$$

Due to the cyclicity of the trace we arrive at the first term in eq. (3.16).

Second summand. We write

$$\begin{aligned} \mathcal{G}_0\mathcal{V}\,\mathcal{G}_0\mathcal{V} &= \begin{pmatrix} G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S_{\mathrm{int}}[\phi, \overline{\phi}] & G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S_{\mathrm{int}}[\phi, \overline{\phi}] \\ -G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\phi} S_{\mathrm{int}}[\phi, \overline{\phi}] & -G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S_{\mathrm{int}}[\phi, \overline{\phi}] \end{pmatrix} \cdot \begin{pmatrix} G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S_{\mathrm{int}}[\phi, \overline{\phi}] & G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S_{\mathrm{int}}[\phi, \overline{\phi}] \\ -G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\phi} S_{\mathrm{int}}[\phi, \overline{\phi}] & -G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S_{\mathrm{int}}[\phi, \overline{\phi}] \end{pmatrix} \\ &\equiv \begin{pmatrix} A + B & \cdots \\ \cdots & C + D \end{pmatrix}, \end{aligned}$$

(the secondary diagonals are irrelevant if one takes the trace and only provide corrections on higher orders of perturbation theory) with

$$
\begin{aligned}
A &= G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S_{\text{int}}[\phi,\overline{\phi}] \cdot G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\phi} S_{\text{int}}[\phi,\overline{\phi}],\\
B &= -G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S_{\text{int}}[\phi,\overline{\phi}] \cdot G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\phi} S_{\text{int}}[\phi,\overline{\phi}],\\
C &= -G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\phi} S_{\text{int}}[\phi,\overline{\phi}] \cdot G_0 \cdot \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}} S_{\text{int}}[\phi,\overline{\phi}],\\
D &= G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S_{\text{int}}[\phi,\overline{\phi}] \cdot G_0^{\mathrm{T}} \cdot \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}} S_{\text{int}}[\phi,\overline{\phi}].
\end{aligned}
$$

Now, one has $\operatorname{Tr} B = \operatorname{Tr} C$. Due to the cyclicity of the trace and $\operatorname{Tr} A = \operatorname{Tr} D$ because of eq. (8.4). Eq. (3.16) is now proven.

Proof 8.12 of Lemma 3.22

This proof is easy on the surface, but difficult when it comes to details.

Preparation: We will first proof that $\exp \pm W_\Lambda[j,\bar{j}]$ can be commuted with any other Graßmann-valued polynomial expressions.

Since

$$
S_\Lambda[\psi,\overline{\psi}] = \sum\!\!\!\!\!\!\int_{x,y} \overline{\psi}(x)\, G^{-1}_{0,\Lambda}(x,y)\, \psi(y) + S_{\text{int}}[\psi,\overline{\psi}]
$$

with an U(1) invariant interaction S_{int}, it follows from Remark 3.15.3, the free energy W must also have an U(1)-invariance.

From this, we can conclude[5] that the free energy W_Λ, expressed by its series expansion, possesses only monomials which have an equal number of j- and $\bar{j}$-fields.

With the expansion of the exponential-function, one follows that $\exp \pm W_\Lambda[j,\bar{j}]$, written by its series expansion, possesses exclusively monomials with even numbers of Graßmann-variables, proving the statement in the preparation.

Now we use the so-called "source-trick" from the literature [12]:

$$
\begin{aligned}
\left(\partial_\Lambda W_\Lambda\right)[j,\bar{j}] &= -\exp\left(W_\Lambda[j,\bar{j}]\right)\partial_\Lambda \exp\left(-W_\Lambda[j,\bar{j}]\right)\\
&= \sum\!\!\!\!\!\!\int_{x,y} \exp\left(W_\Lambda[j,\bar{j}]\right)\int \mathscr{D}(\psi,\overline{\psi})\,\overline{\psi}(x)\,\left(\partial_\Lambda G^{-1}_{0,\Lambda}\right)(x,y)\,\psi(y)\,\exp\left(-S_\Lambda + \sum\!\!\!\!\!\!\int \bar{j}\psi + \overline{\psi} j\right)\\
&= -\sum\!\!\!\!\!\!\int_{x,y} \exp\left(W_\Lambda[j,\bar{j}]\right)\frac{\vec{\delta}}{\delta j(x)}\left(\partial_\Lambda G^{-1}_{0,\Lambda}\right)(x,y)\frac{\vec{\delta}}{\delta\bar{j}(y)}\exp\left(-W_\Lambda[j,\bar{j}]\right)\\
&= \sum\!\!\!\!\!\!\int_{x,y} -\left(\frac{\vec{\delta}}{\delta j(x)}W_\Lambda[j,\bar{j}]\right)\left(\partial_\Lambda G^{-1}_{0,\Lambda}\right)(x,y)\left(\frac{\vec{\delta}}{\delta\bar{j}(y)}W_\Lambda[j,\bar{j}]\right) + \left(\partial_\Lambda G^{-1}_{0,\Lambda}\right)(x,y)\frac{\vec{\delta}^2}{\delta j(x)\delta\bar{j}(y)}W_\Lambda[j,\bar{j}].
\end{aligned}
$$

[5] See, for example, [7] for details.

Using Lemma 3.20 twice gives for the second term

$$\begin{aligned}\sum\!\!\!\!\!\!\int_{x,y}\left(\partial_\Lambda G_{0,\Lambda}^{-1}\right)(x,y)\frac{\vec{\delta}^2}{\delta j(x)\delta\bar{j}(y)}W_\Lambda[j,\bar{j}] &= \sum\!\!\!\!\!\!\int_{x,y}\left(\partial_\Lambda G_{0,\Lambda}^{-1}\right)(x,y)\frac{\delta^2}{\delta\bar{j}(y)\delta j(x)}W_\Lambda[j,\bar{j}]\\ &= \mathrm{Tr}\left[\partial_\Lambda G_{0,\Lambda}^{-1}\cdot\frac{\delta^2}{\delta\bar{j}\,\delta j}W_\Lambda[j,\bar{j}]\right]\end{aligned}$$

and completes the proof.

Proof 8.13 of Lemma 3.23

We now show

$$\left(\begin{pmatrix}\frac{-\delta^2}{\delta\bar{j}\,\delta j}W_\Lambda & \frac{\delta^2}{\delta\bar{j}\,\delta\bar{j}}W_\Lambda\\ \frac{\delta^2}{\delta j\,\delta j}W_\Lambda & \frac{-\delta^2}{\delta j\,\delta\bar{j}}W_\Lambda\end{pmatrix}\cdot\begin{pmatrix}\frac{\delta^2}{\delta\bar{\phi}\,\delta\phi}\Gamma_\Lambda & \frac{\delta^2}{\delta\bar{\phi}\,\delta\bar{\phi}}\Gamma_\Lambda\\ \frac{\delta^2}{\delta\phi\,\delta\phi}\Gamma_\Lambda & \frac{\delta^2}{\delta\phi\,\delta\bar{\phi}}\Gamma_\Lambda\end{pmatrix}\right)(x',x)=\begin{pmatrix}\delta(x'-x) & 0\\ 0 & \delta(x'-x)\end{pmatrix}$$

by applying the chain rule:

With the expansion $j(x)=\sum\!\!\!\!\!\!\int_y \phi(y)\left(\frac{\vec{\delta}}{\delta\phi(y)}j(x)\right)+\bar{\phi}(y)\left(\frac{\vec{\delta}}{\delta\bar{\phi}(y)}j(x)\right)$ one immediately arrives at

$$\begin{aligned}\delta(x-x') &= \frac{\vec{\delta}}{\delta j(x')}j(x)\\ &= \sum\!\!\!\!\!\!\int_y\left(\frac{\vec{\delta}}{\delta j(x')}\phi(y)\right)\left(\frac{\vec{\delta}}{\delta\phi(y)}j(x)\right)+\left(\frac{\vec{\delta}}{\delta j(x')}\bar{\phi}(y)\right)\left(\frac{\vec{\delta}}{\delta\bar{\phi}(y)}j(x)\right)\\ &= \sum\!\!\!\!\!\!\int_y\left(\frac{\delta^2}{\delta j(x')\delta\bar{j}(y)}W_\Lambda[j,\bar{j}]\right)\left(\frac{-\delta^2}{\delta\phi(y)\delta\bar{\phi}(x)}\Gamma_\Lambda[\phi,\bar{\phi}]\right)+\left(\frac{-\delta^2}{\delta j(x')\delta j(y)}W_\Lambda[j,\bar{j}]\right)\left(\frac{-\delta^2}{\delta\bar{\phi}(y)\delta\bar{\phi}(x)}\Gamma_\Lambda[\phi,\bar{\phi}]\right).\end{aligned}$$

Further, it follows

$$\begin{aligned}0 &= \frac{\vec{\delta}}{\delta\bar{j}(x')}j(x)\\ &= \sum\!\!\!\!\!\!\int_y\left(\frac{\vec{\delta}}{\delta\bar{j}(x')}\phi(y)\right)\left(\frac{\vec{\delta}}{\delta\phi(y)}j(x)\right)+\left(\frac{\vec{\delta}}{\delta\bar{j}(x')}\bar{\phi}(y)\right)\left(\frac{\vec{\delta}}{\delta\bar{\phi}(y)}j(x)\right)\\ &= \sum\!\!\!\!\!\!\int_y\left(\frac{\delta^2}{\delta\bar{j}(x')\delta\bar{j}(y)}W_\Lambda[j,\bar{j}]\right)\left(\frac{-\delta^2}{\delta\phi(y)\delta\bar{\phi}(x)}\Gamma_\Lambda[\phi,\bar{\phi}]\right)+\left(\frac{-\delta^2}{\delta\bar{j}(x')\delta j(y)}W_\Lambda[j,\bar{j}]\right)\left(\frac{-\delta^2}{\delta\bar{\phi}(y)\delta\bar{\phi}(x)}\Gamma_\Lambda[\phi,\bar{\phi}]\right).\end{aligned}$$

Analogously, one gets

$$
\begin{aligned}
0 &= \frac{\vec{\delta}}{\delta j(x')}\bar{j}(x) \\
&= \sum\!\!\!\!\!\!\int_y \left(\frac{\vec{\delta}}{\delta j(x')}\phi(y)\right)\left(\frac{\vec{\delta}}{\delta\phi(y)}\bar{j}(x)\right) + \left(\frac{\vec{\delta}}{\delta j(x')}\bar{\phi}(y)\right)\left(\frac{\vec{\delta}}{\delta\bar{\phi}(y)}\bar{j}(x)\right) \\
&= \sum\!\!\!\!\!\!\int_y \left(\frac{\delta^2}{\delta j(x')\delta\bar{j}(y)}W_\Lambda[j,\bar{j}]\right)\left(\frac{\delta^2}{\delta\phi(y)\delta\phi(x)}\Gamma_\Lambda[\phi,\bar{\phi}]\right) + \left(\frac{-\delta^2}{\delta j(x')\delta j(y)}W_\Lambda[j,\bar{j}]\right)\left(\frac{\delta^2}{\delta\bar{\phi}(y)\delta\phi(x)}\Gamma_\Lambda[\phi,\bar{\phi}]\right).
\end{aligned}
$$

And last but not least, one has

$$
\begin{aligned}
\delta(x-x') &= \frac{\vec{\delta}}{\delta\bar{j}(x')}\bar{j}(x) \\
&= \sum\!\!\!\!\!\!\int_y \left(\frac{\vec{\delta}}{\delta\bar{j}(x')}\phi(y)\right)\left(\frac{\vec{\delta}}{\delta\phi(y)}\bar{j}(x)\right) + \left(\frac{\vec{\delta}}{\delta\bar{j}(x')}\bar{\phi}(y)\right)\left(\frac{\vec{\delta}}{\delta\bar{\phi}(y)}\bar{j}(x)\right) \\
&= \sum\!\!\!\!\!\!\int_y \left(\frac{\delta^2}{\delta\bar{j}(x')\delta\bar{j}(y)}W_\Lambda[j,\bar{j}]\right)\left(\frac{\delta^2}{\delta\phi(y)\delta\phi(x)}\Gamma_\Lambda[\phi,\bar{\phi}]\right) + \left(\frac{-\delta^2}{\delta\bar{j}(x')\delta j(y)}W_\Lambda[j,\bar{j}]\right)\left(\frac{\delta^2}{\delta\bar{\phi}(y)\delta\phi(x)}\Gamma_\Lambda[\phi,\bar{\phi}]\right).
\end{aligned}
$$

Proof 8.14 of Theorem 3.24

The important thing in this calculation is to consider the Λ-dependence of the sources j und $\bar{j}$: First it follows from the linearity of derivatives

$$
\begin{aligned}
\partial_\Lambda\Gamma_\Lambda[\phi,\bar{\phi}] &= \partial_\Lambda W_\Lambda[j,\bar{j}] + \partial_\Lambda \sum\!\!\!\!\!\!\int \left(\bar{j}\phi + \bar{\phi}j\right) \\
&= \partial_\Lambda W_\Lambda[j,\bar{j}] + \sum\!\!\!\!\!\!\int \left((\partial_\Lambda\bar{j})\,\phi + \bar{\phi}\,\partial_\Lambda j\right).
\end{aligned}
$$

We now concentrate on the first term. With eq. (3.5) and Remark 3.22.2 we get

$$
\begin{aligned}
\partial_\Lambda W_\Lambda[j,\bar{j}] &= -\partial_\Lambda \log Z_\Lambda[j,\bar{j}] \\
&= -\left(\frac{\partial_\Lambda Z_\Lambda}{Z_\Lambda}\right)[j,\bar{j}] - \sum\!\!\!\!\!\!\int \left((\partial_\Lambda\bar{j})\,\langle\psi\rangle_{j,\bar{j},\Lambda} + \langle\bar{\psi}\rangle_{j,\bar{j},\Lambda}\,\partial_\Lambda j\right) \\
&= +\left(\frac{\partial}{\partial\Lambda}W_\Lambda\right)[j,\bar{j}] - \sum\!\!\!\!\!\!\int \left((\partial_\Lambda\bar{j})\,\phi + \bar{\phi}\,\partial_\Lambda j\right)
\end{aligned}
$$

Thus, the derivatives of the sources cancel out and using the remaining identities in Notation 3.10 and the flow equation in Lemma 3.22, we obtain

$$\begin{aligned}\partial_\Lambda \Gamma_\Lambda[\phi,\overline{\phi}] &= +\left(\frac{\partial}{\partial\Lambda} W_\Lambda\right)[j,\bar{j}] \\ &= \mathrm{Tr}\left(\partial_\Lambda G_{0,\Lambda}^{-1}\cdot\frac{\delta^2}{\delta\bar{j}\delta j}W_\Lambda[j,\bar{j}]\right) - \sum\!\!\!\!\!\!\int_{x,y}\left(\frac{\vec{\delta}}{\delta j(x)}W_\Lambda[j,\bar{j}]\right)\left(\partial_\Lambda G_{0,\Lambda}^{-1}\right)(x,y)\left(\frac{\vec{\delta}}{\delta\bar{j}(y)}W_\Lambda[j,\bar{j}]\right) \\ &= \mathrm{Tr}\left(\partial_\Lambda G_{0,\Lambda}^{-1}\cdot\frac{\delta^2}{\delta\bar{j}\delta j}W_\Lambda[j,\bar{j}]\right) + \sum\!\!\!\!\!\!\int_{x,y}\overline{\phi}(x)\left(\partial_\Lambda G_{0,\Lambda}^{-1}\right)(x,y)\,\phi(y) = \dots\end{aligned}$$

We now use Lemma 3.23 to see the equation

$$-\frac{\delta^2}{\delta\bar{j}\delta j}W_\Lambda[j,\bar{j}] = \begin{pmatrix} \frac{\delta^2}{\delta\overline{\phi}\delta\phi}\Gamma_\Lambda[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\delta\overline{\phi}}\Gamma_\Lambda[\phi,\overline{\phi}] \\ \frac{\delta^2}{\delta\phi\delta\phi}\Gamma_\Lambda[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\delta\overline{\phi}}\Gamma_\Lambda[\phi,\overline{\phi}] \end{pmatrix}^{-1}_{1,1}.$$

Now, we combine these results, use the cyclicity of the trace and the antisymmetry of the corresponding operators, and thus obtain

$$\dots = -\frac{1}{2}\,\mathrm{Tr}\left(\frac{\begin{pmatrix}\partial_\Lambda G_{0,\Lambda}^{-1} & 0\\ 0 & -\partial_\Lambda G_{0,\Lambda}^{-\mathrm{T}}\end{pmatrix}}{\begin{pmatrix} \frac{\delta^2}{\delta\overline{\phi}\delta\phi}\Gamma_\Lambda[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\delta\overline{\phi}}\Gamma_\Lambda[\phi,\overline{\phi}] \\ \frac{\delta^2}{\delta\phi\delta\phi}\Gamma_\Lambda[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\delta\overline{\phi}}\Gamma_\Lambda[\phi,\overline{\phi}] \end{pmatrix}}\right) + \sum\!\!\!\!\!\!\int_{x,y}\overline{\phi}(x)\left(\partial_\Lambda G_{0,\Lambda}^{-1}\right)(x,y)\,\phi(y),$$

Proof 8.15 of Theorem 3.26

First we calculate using the Neumann-series (which is valid in the present case (?!))

$$\begin{aligned}\left(\boldsymbol{\delta}^2\Gamma_\Lambda[\phi,\overline{\phi}]\right)^{-1} &= \left(\boldsymbol{G}_\Lambda^{-1} - \boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}]\right)^{-1} \\ &= \left(1 - \boldsymbol{G}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}]\right)^{-1}\cdot\boldsymbol{G}_\Lambda \\ &= \left(1 + \boldsymbol{G}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}] + \boldsymbol{G}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}]\cdot\boldsymbol{G}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}] + \dots\right)\cdot\boldsymbol{G}_\Lambda\end{aligned}$$

where we used $(AB)^{-1} = B^{-1}A^{-1}$ and the geometric series. Then by substitution into the Wetterich-equation

$$\partial_\Lambda\Gamma_\Lambda[\phi,\overline{\phi}] = -\frac{1}{2}\,\mathrm{Tr}\left(\frac{\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1}}{\boldsymbol{\delta}^2\Gamma_\Lambda[\phi,\overline{\phi}]}\right) + \sum\!\!\!\!\!\!\int_{x,y}\overline{\phi}(x)\,\partial_\Lambda G_{0,\Lambda}^{-1}(x,y)\,\phi(y),$$

it follows for the first term of the r.h.s. with the cyclicity and linearity of the trace

$$\mathrm{Tr}\left(\frac{\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1}}{\boldsymbol{\delta}^2\Gamma_\Lambda[\phi,\overline{\phi}]}\right) = \mathrm{Tr}\Bigg(\left(\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1}\right)\cdot\left(1+\boldsymbol{G}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}]+\boldsymbol{G}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}]\cdot\boldsymbol{G}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}]+\ldots\right)\cdot\boldsymbol{G}_\Lambda\Bigg)$$
$$= \mathrm{Tr}\left(\boldsymbol{G}_\Lambda\cdot\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1}\right)+\mathrm{Tr}\Bigg(\boldsymbol{S}_\Lambda\cdot\left(\boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}]+\boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}]\cdot\boldsymbol{G}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda[\phi,\overline{\phi}]+\ldots\right)\Bigg).$$

If we now use fundamental identities of the trace again (invariance under transposition, cyclicity and linearity) and sort the terms in the Wetterich-equation, we get the desired result.

Proof 8.16 of Corollary 3.31

From Theorem 3.15 it immediately follows that the choice of effective action satisfies the required symmetry properties. Also eq. (3.18) is obviously satisfied with this choice of regulator; A typical choice[6] for a regularized Heaviside-function would be

$$(f(x)-\Lambda) \quad\mapsto\quad \theta_\epsilon(f(x)-\Lambda) \;=\; \frac{1}{\pi}\arctan\left(\frac{f(x)-\Lambda}{\epsilon}\right)+\frac{1}{2}.$$

Then, one has for $\Lambda = 0$

$$\left(\theta_\epsilon(f(x)-0)\right)^{-1} = \left(\frac{1}{\pi}\arctan\left(\frac{f(x)}{\epsilon}\right)+\frac{1}{2}\right)^{-1} \to 1 \quad \text{for} \quad \epsilon\to 0.$$

Similarly, for $f(x) \ll \Lambda$ the limiting behavior is

$$\left(\theta_\epsilon(f(x)-\Lambda)\right)^{-1} \approx \left(\frac{1}{\pi}\arctan\left(\frac{-\Lambda}{\epsilon}\right)+\frac{1}{2}\right)^{-1} \to +\infty \quad \text{for} \quad \epsilon\to 0.$$

Proof 8.17 of Corollary 3.32

The first identity is clear. The second identity follows with Notation 3.25. For the third identity we have, after introducing the abbreviation

$$\mathrm{Mat}[\theta_{\Lambda,\epsilon}](x,y) = \delta(x-y)\,\theta_\epsilon\left(f(x)-\Lambda\right) \equiv \delta(x-y)g_\Lambda(x),$$

the identities

$$\boldsymbol{G}_{0,\Lambda}(x,y) = \sum\!\!\!\!\!\!\int_\alpha \theta_{\Lambda,\epsilon}(x,\alpha)\,\boldsymbol{G}_0(\alpha,y) = g_\Lambda(x)\,\boldsymbol{G}_0(x,y),$$

$$\left(\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1}\right)(x,y) = \sum\!\!\!\!\!\!\int_\alpha \boldsymbol{G}_0^{-1}(x,\alpha)\,\partial_\Lambda\theta_{\Lambda,\epsilon}^{-1}(\alpha,y) = -\boldsymbol{G}_0^{-1}(x,y)\,\frac{\partial_\Lambda g_\Lambda(y)}{g_\Lambda(y)^2}.$$

[6] This function converges (as a Schwartz-distribution) to a Heaviside-function. Correspondingly its derivative converges (again as a Schwartz-distribution) to a Dirac-delta-function (up to a factor).

These two identities can be combined to

$$\begin{aligned}\left(\left(\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1}\right)\cdot \boldsymbol{G}_{0,\Lambda}\right)(x,y) &= \sum\!\!\!\!\!\!\int_\alpha \left(\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1}\right)(x,\alpha)\cdot \boldsymbol{G}_{0,\Lambda}(\alpha,y)\\ &= -\sum\!\!\!\!\!\!\int_\alpha \boldsymbol{G}_0^{-1}(x,\alpha)\,\frac{\partial_\Lambda g_\Lambda(\alpha)}{g_\Lambda(\alpha)}\boldsymbol{G}_0(\alpha,y).\end{aligned}$$

In total one gets

$$\begin{aligned}\boldsymbol{S}_\Lambda(x,y) &= \left(\boldsymbol{G}_{0,\Lambda}\cdot\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1}\cdot \boldsymbol{G}_{0,\Lambda}\right)(x,y)\\ &= \sum\!\!\!\!\!\!\int_\alpha \boldsymbol{G}_\Lambda(x,\alpha)\cdot\left(\partial_\Lambda \boldsymbol{G}_{0,\Lambda}^{-1}\cdot \boldsymbol{G}_\Lambda\right)(\alpha,y)\\ &= -\sum\!\!\!\!\!\!\int_\alpha g_\Lambda(x)\,\boldsymbol{G}_0(x,\alpha)\cdot\sum\!\!\!\!\!\!\int_\beta \boldsymbol{G}_0^{-1}(\alpha,\beta)\,\frac{\partial_\Lambda g_\Lambda(\beta)}{g_\Lambda(\beta)}\boldsymbol{G}_0(\beta,y)\\ &= -\sum\!\!\!\!\!\!\int_\beta g_\Lambda(x)\delta(x-\beta)\frac{\partial_\Lambda g_\Lambda(\beta)}{g_\Lambda(\beta)}\boldsymbol{G}_0(\beta,y)\\ &= -\partial_\Lambda g_\Lambda(x)\boldsymbol{G}_0(x,y)\\ &= -\partial_\Lambda \boldsymbol{G}_{0,\Lambda}(x,y),\end{aligned}$$

implying (referring to Notation 3.25)

$$\boldsymbol{S}_\Lambda = \begin{pmatrix}-\partial_\Lambda G_{0,\Lambda} & 0\\ 0 & +\partial_\Lambda G_{0,\Lambda}^{\mathrm{T}}\end{pmatrix} = \begin{pmatrix}S_\Lambda & 0\\ 0 & -S_\Lambda^{\mathrm{T}}\end{pmatrix}, \tag{8.5}$$

the announced result.

Proof 8.18 of Theorem 3.33

In order to calculate

$$-\frac{1}{2}\,\mathrm{Tr}\left(\boldsymbol{S}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda\cdot\boldsymbol{G}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda\right) \quad = \quad +\frac{1}{2}\,\mathrm{Tr}\left(\partial_\Lambda \boldsymbol{G}_{0,\Lambda}\cdot\boldsymbol{\delta}^2\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]\cdot\boldsymbol{G}_{0,\Lambda}\cdot\boldsymbol{\delta}^2\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]\right)$$

we first take the trace w.r.t. ϕ- and $\overline{\phi}$-components, by multiplying the matrices. One gets

$$\begin{aligned}\partial_\Lambda \boldsymbol{G}_{0,\Lambda}\cdot\boldsymbol{\delta}^2\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}] &= \begin{pmatrix}\partial_\Lambda G_{0,\Lambda} & 0\\ 0 & -\partial_\Lambda G_{0,\Lambda}^{\mathrm{T}}\end{pmatrix}\cdot\begin{pmatrix}\frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]\\ \frac{\delta^2}{\delta\phi\,\delta\phi}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}] & \frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]\end{pmatrix}\\ &= \begin{pmatrix}\partial_\Lambda G_{0,\Lambda}\cdot\frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}] & \partial_\Lambda G_{0,\Lambda}\cdot\frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]\\ -\partial_\Lambda G_{0,\Lambda}^{\mathrm{T}}\cdot\frac{\delta^2}{\delta\phi\,\delta\phi}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}] & -\partial_\Lambda G_{0,\Lambda}^{\mathrm{T}}\cdot\frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]\end{pmatrix},\end{aligned}$$

and analogously one gets

$$\boldsymbol{G}_{0,\Lambda}\cdot\boldsymbol{\delta}^2\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]=\begin{pmatrix} G_{0,\Lambda}\cdot\frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}] & G_{0,\Lambda}\cdot\frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}] \\ -G^{\mathrm{T}}_{0,\Lambda}\cdot\frac{\delta^2}{\delta\phi\,\delta\phi}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}] & -G^{\mathrm{T}}_{0,\Lambda}\cdot\frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}] \end{pmatrix}.$$

With this one arrives at

$$\partial_\Lambda G_{0,\Lambda}\cdot\boldsymbol{\mathcal{V}}_\Lambda\cdot\boldsymbol{G}_\Lambda\cdot\boldsymbol{\mathcal{V}}_\Lambda=\begin{pmatrix} A+B & \cdots \\ \cdots & C+D \end{pmatrix},$$

(the secondary diagonals are irrelevant if one takes the trace and only provide corrections on higher orders of perturbation theory) with

$$\begin{aligned}
A &= \partial_\Lambda G_{0,\Lambda}\cdot\frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]\cdot G_{0,\Lambda}\cdot\frac{\delta^2}{\delta\overline{\phi}\,\delta\phi}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}],\\
B &= -\partial_\Lambda G_{0,\Lambda}\cdot\frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]\cdot G^{\mathrm{T}}_{0,\Lambda}\cdot\frac{\delta^2}{\delta\phi\,\delta\phi}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}],\\
C &= -\partial_\Lambda G^{\mathrm{T}}_{0,\Lambda}\cdot\frac{\delta^2}{\delta\phi\,\delta\phi}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]\cdot G_{0,\Lambda}\cdot\frac{\delta^2}{\delta\overline{\phi}\,\delta\overline{\phi}}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}],\\
D &= \partial_\Lambda G^{\mathrm{T}}_{0,\Lambda}\cdot\frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}]\cdot G^{\mathrm{T}}_{0,\Lambda}\cdot\frac{\delta^2}{\delta\phi\,\delta\overline{\phi}}\Gamma_{\Lambda,\mathrm{int}}[\phi,\overline{\phi}].
\end{aligned}$$

Now, one has $\mathrm{Tr}\,B=\mathrm{Tr}\,C$ due to the cyclicity of the trace and further one has $\mathrm{Tr}\,A=\mathrm{Tr}\,D$ because of eq. (8.4).

8.1.2 Proofs to Chapter 4.

Proof 8.19 of the footnote on page 33

In the template [15] the operators l and r are introduced as

$$\begin{aligned}
l &:= V_L a := \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\Sigma_y}(E^{11}t+E^{22})a,\\
r &:= V_R a := \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\Sigma_y}(E^{11}+E^{22}t)a,
\end{aligned}$$

where

- t denotes the translation-matrix, i.e. $(\mathrm{mat}(t))_{i,j}=\delta_{i+1,j}$,
- $a\equiv(a_1\,,\,a_2\,,\,\ldots\,,a_M)^{\mathrm{T}}$,
- $a_j\equiv(a_{1,j}\,,\,a_{2,j})^{\mathrm{T}}$,
- $\Sigma_j\equiv\sigma_j\otimes 1$ with $j\in\{x,y,z\}$,
- $\left(\mathrm{mat}(E^{i,j})\right)_{kl}\equiv\delta_{i,k}\delta_{j,l}$.

Preparations:

- For l:

$$(E^{1,1}\, t\, a)_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{1,j+1} \\ a_{2,j+1} \end{pmatrix} = \begin{pmatrix} a_{1,j+1} \\ 0 \end{pmatrix},$$
$$(E^{2,2}\, a)_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{1,j} \\ a_{2,j} \end{pmatrix} = \begin{pmatrix} 0 \\ a_{2,j} \end{pmatrix}.$$

- For r:

$$(E^{1,1}\, a)_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{1,j} \\ a_{2,j} \end{pmatrix} = \begin{pmatrix} a_{1,j} \\ 0 \end{pmatrix},$$
$$(E^{2,2}\, t\, a)_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{1,j+1} \\ a_{2,j+1} \end{pmatrix} = \begin{pmatrix} 0 \\ a_{2,j+1} \end{pmatrix}.$$

- For both operators:

$$\begin{aligned}
\exp\left(\mathrm{i}\frac{\pi}{4}\Sigma_y\right) &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\mathrm{i}\frac{\pi}{4}\sigma_y \otimes \mathrm{Id}\right)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{(2k)!} \left(\mathrm{i}\frac{\pi}{4}\right)^{2k} + \frac{1}{(2k+1)!} \left(\mathrm{i}\frac{\pi}{4}\right)^{2k+1} \sigma_y \otimes \mathrm{Id} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{\pi}{4}\right)^{2k} + \mathrm{i}\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left(\frac{\pi}{4}\right)^{2k+1}\right) \sigma_y \otimes \mathrm{Id} \\
&= \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right) \sigma_y \otimes \mathrm{Id} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \otimes \mathrm{Id}\,.
\end{aligned}$$

With this one gets …

- for the concrete expression of l:

$$\begin{aligned}
l_j = (V_L a)_j &= \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\Sigma_y} \left(E^{1,1}\, t + E^{2,2}\right) a_j \\
&= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \otimes \mathrm{Id}\right) \left(\begin{pmatrix} a_{1,j+1} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_{2,j} \end{pmatrix}\right) \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} a_{1,j+1} + a_{2,j} \\ -a_{1,j+1} + a_{2,j} \end{pmatrix}
\end{aligned}$$

and therefore

$$l_{\nu,j} = \frac{1}{\sqrt{2}} \left((-1)^{\nu+1} a_{1,j+1} + a_{2,j}\right).$$

- for the concrete expression of r:

$$\begin{aligned} r_j = (V_R a)_j &= \mathrm{e}^{\mathrm{i}\frac{\pi}{4}\Sigma_y}\left(E^{1,1} + E^{2,2}\,t\right) a_j \\ &= \frac{1}{\sqrt{2}}\left(\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \otimes \mathrm{Id}\right)\left(\begin{pmatrix} a_{1,j} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_{2,j+1} \end{pmatrix}\right) \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} a_{1,j} + a_{2,j+1} \\ -a_{1,j} + a_{2,j} \end{pmatrix}, \end{aligned}$$

  and therefore

$$r_{\nu,j} = \frac{1}{\sqrt{2}}\left((-1)^{\nu+1} a_{1,j} + a_{2,j+1}\right).$$

Proof 8.20 of Theorem 4.3 The equations

$$\{l_{\nu,j}, l_{\nu',j'}\} = \{l^\dagger_{\nu,j}, l^\dagger_{\nu',j'}\} = \{r_{\nu,j}, r_{\nu',j'}\} = \{r^\dagger_{\nu,j}, r^\dagger_{\nu',j'}\} = 0$$

are trivial. Further one has

$$\begin{aligned} \{l_{\nu,j}, l^\dagger_{\nu',j'}\} &= \frac{1}{2}\{(-1)^{\nu+1} a_{1,j+1} + a_{2,j}\,,\, (-1)^{\nu'+1} a^\dagger_{1,j'+1} + a^\dagger_{2,j'}\} \\ &= \frac{1}{2}\delta_{j,j'}\left((-1)^{\nu+\nu'} + 1\right) \\ &= \delta_{\nu,\nu'}\delta_{j,j'} \end{aligned}$$

and analogously one has

$$\{r_{\nu,j}, r^\dagger_{\nu',j'}\} = \delta_{\nu,\nu'}\delta_{j,j'}.$$

Proof 8.21 of Corollary 4.4.

The Hamiltonian H_{L}

Using definition 4.2, we obtain for the first Hamiltonian

$$H_{\mathrm{L}} = \sum_j \begin{pmatrix} l_{1,j} \\ l_{2,j} \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} l_{1,j} \\ l_{2,j} \end{pmatrix} = \sum_j l^\dagger_{1,j} l_{1,j} - l^\dagger_{2,j} l_{2,j}.$$

We examine both contributions separately; for $\nu \in \{1, 2\}$ we then obtain

$$\begin{aligned} \sum_j l^\dagger_{\nu,j} l_{\nu,j} &= \frac{1}{2}\sum_j \left((-1)^{\nu+1} a^\dagger_{1,j+1} + a^\dagger_{2,j}\right)\left((-1)^{\nu+1} a_{1,j+1} + a_{2,j}\right) \\ &= \frac{1}{2}\sum_j \left(a^\dagger_{1,j+1} a_{1,j+1} + (-1)^{\nu+1}\left(a^\dagger_{1,j+1} a_{2,j} + a^\dagger_{2,j} a_{1,j+1}\right) + a^\dagger_{2,j} a_{2,j}\right) \\ &= \dots \end{aligned}$$

which can, with the Fourier-transform[7]

$$a_{\nu,j} = \frac{1}{\sqrt{M}} \sum_q \mathrm{e}^{-\mathrm{i}\,q\,x_j} a_{\nu,q},$$

be further simplified to

$$\begin{aligned}\ldots &= \frac{1}{2M} \sum_{j,p,q} \left(\mathrm{e}^{\mathrm{i}\,q\,x_{j+1}} a^\dagger_{1,q} \mathrm{e}^{-\mathrm{i}\,p\,x_{j+1}} a_{1,p} + (-1)^{\nu+1} \left(\mathrm{e}^{\mathrm{i}\,q\,x_{j+1}} a^\dagger_{1,q} \mathrm{e}^{-\mathrm{i}\,p\,x_j} a_{2,p} + \mathrm{e}^{\mathrm{i}\,q\,x_j} a^\dagger_{2,q} \mathrm{e}^{-\mathrm{i}\,p\,x_{j+1}} a_{1,p} \right) + \mathrm{e}^{\mathrm{i}\,q\,x_j} a^\dagger_{2,q} \mathrm{e}^{-\mathrm{i}\,p\,x_j} a_{2,p} \right) \\ &= \frac{1}{2} \sum_q \left(a^\dagger_{1,q} a_{1,q} + (-1)^{\nu+1} \left(\mathrm{e}^{\mathrm{i}\,q\,a} a^\dagger_{1,q} a_{2,q} + \mathrm{e}^{-\mathrm{i}\,q\,a} a^\dagger_{2,q} a_{1,q} \right) + a^\dagger_{2,q} a_{2,q} \right).\end{aligned}$$

This then provides in total

$$H_\mathrm{L} = \sum_j l^\dagger_{1,j} l_{1,j} - l^\dagger_{2,j} l_{2,j} = \sum_q \left(\mathrm{e}^{\mathrm{i}\,q\,a} a^\dagger_{1,q} a_{2,q} + \mathrm{e}^{-\mathrm{i}\,q\,a} a^\dagger_{2,q} a_{1,q} \right) = \sum_q \begin{pmatrix} a_{1,q} \\ a_{2,q} \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathrm{e}^{+\mathrm{i}\,q\,a} \\ \mathrm{e}^{-\mathrm{i}\,q\,a} & 0 \end{pmatrix} \begin{pmatrix} a_{1,q} \\ a_{2,q} \end{pmatrix}.$$

The Hamiltonian H_R

Quite analogously one obtains again with definition 4.2 for the other Hamiltonian

$$H_\mathrm{R} = \sum_j \begin{pmatrix} r_{1,j} \\ r_{2,j} \end{pmatrix}^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} r_{1,j} \\ r_{2,j} \end{pmatrix} = \sum_j r^\dagger_{1,j} r_{1,j} - r^\dagger_{2,j} r_{2,j}.$$

We again consider both summands individually and calculate for $\nu \in \{1, 2\}$

$$\begin{aligned}\sum_j r^\dagger_{\nu,j} r_{\nu,j} &= \frac{1}{2} \sum_j \left((-1)^{\nu+1} a^\dagger_{1,j} + a^\dagger_{2,j+1} \right) \left((-1)^{\nu+1} a_{1,j} + a_{2,j+1} \right) \\ &= \frac{1}{2} \sum_j \left(a^\dagger_{1,j} a_{1,j} + (-1)^{\nu+1} (a^\dagger_{1,j} a_{2,j+1} + a^\dagger_{2,j+1} a_{1,j}) + a^\dagger_{2,j+1} a_{2,j+1} \right) \\ &= \frac{1}{2} \sum_q \left(a^\dagger_{1,q} a_{1,q} + (-1)^{\nu+1} (\mathrm{e}^{-\mathrm{i}qa} a^\dagger_{1,q} a_{2,q} + \mathrm{e}^{\mathrm{i}qa} a^\dagger_{2,q} a_{1,q}) + a^\dagger_{2,q} a_{2,q} \right).\end{aligned}$$

And with this we finally get

$$H_\mathrm{R} = \sum_q e^{-iqa} a^\dagger_{1,q} a_{2,q} + e^{iqa} a^\dagger_{2,q} a_{1,q} = \sum_q \begin{pmatrix} a_{1,q} \\ a_{2,q} \end{pmatrix}^\dagger \begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\,q\,a} \\ \mathrm{e}^{+\mathrm{i}\,q\,a} & 0 \end{pmatrix} \begin{pmatrix} a_{1,q} \\ a_{2,q} \end{pmatrix}.$$

[7] Cf. Theorem 4.9 for the prefactor-convention.

Proof 8.22 of Theorem 4.6

First we notice the decompositions

$$H_\bullet(q) = \begin{cases} \cos(qa)\cdot\sigma_x - \sin(qa)\cdot\sigma_y & \text{for } \bullet = \mathrm{L} \\ \cos(qa)\cdot\sigma_x + \sin(qa)\cdot\sigma_y & \text{for } \bullet = \mathrm{R} \end{cases},$$

which give with Notation 4.5

$$\vec{h}_\mathrm{L}(q) = \begin{pmatrix} \cos(qa) \\ -\sin(qa) \\ 0 \end{pmatrix}, \qquad \vec{h}_\mathrm{R}(q) = \begin{pmatrix} \cos(qa) \\ \sin(qa) \\ 0 \end{pmatrix}$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}k}\vec{h}_\mathrm{L}(q) = a\begin{pmatrix} -\sin(qa) \\ -\cos(qa) \\ 0 \end{pmatrix}, \qquad \frac{\mathrm{d}}{\mathrm{d}k}\vec{h}_\mathrm{R}(q) = a\begin{pmatrix} -\sin(qa) \\ \cos(qa) \\ 0 \end{pmatrix}.$$

This immediately gives

$$\vec{h}_\bullet(q) \times \frac{\mathrm{d}}{\mathrm{d}q}\vec{h}_\bullet(q) = \begin{cases} -a\vec{e}_z & \text{for } \bullet = \mathrm{L} \\ +a\vec{e}_z & \text{for } \bullet = \mathrm{R} \end{cases}.$$

Substituting in eq. (4.6) then yields

$$W_\mathrm{L} = -\frac{a}{2\pi}\int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}} \mathrm{d}q\, 1 = -1, \qquad W_\mathrm{R} = +\frac{a}{2\pi}\int_{-\frac{\pi}{a}}^{+\frac{\pi}{a}} \mathrm{d}q\, 1 = +1.$$

Proof 8.23 of Theorem 4.8

The equations

$$\{x_{\nu,j}, x_{\nu',j'}\} = \{x^\dagger_{\nu,j}, x^\dagger_{\nu',j'}\} = 0$$

are trivial again.

As preparations we compute

- $\{l_{\nu,j}\,,\, r^\dagger_{\nu',j'}\} = \frac{1}{2}\{(-1)^{\nu+1}a_{1,j+1} + a_{2,j}\,,\, (-1)^{\nu'+1}a^\dagger_{1,j'} + a^\dagger_{2,j'+1}\} = \frac{1}{2}\left((-1)^{\nu+\nu'}\delta_{j+1,j'} + \delta_{j,j'+1}\right)$,
- $\{r_{\nu,j}\,,\, l^\dagger_{\nu',j'}\} = \frac{1}{2}\{(-1)^{\nu+1}a_{1,j} + a_{2,j+1}\,,\, (-1)^{\nu'+1}a^\dagger_{1,j'+1} + a^\dagger_{2,j'}\} = \frac{1}{2}\left((-1)^{\nu+\nu'}\delta_{j,j'+1} + \delta_{j+1,j'}\right)$.

Then it follows with Remark 4.3

$$\begin{aligned}\{x_{\nu,j}, x^\dagger_{\nu',j'}\} &= \{l_{\nu,j}\cos\theta + r_{\nu,j}\sin\theta\,,\, l^\dagger_{\nu',j'}\cos\theta + r^\dagger_{\nu',j'}\sin\theta\} \\ &= \delta_{\nu,\nu'}\delta_{j,j'} + \sin\theta\cos\theta\left(\{l_{\nu,j}, r^\dagger_{\nu',j'}\} + \{r_{\nu,j}, l^\dagger_{\nu',j'}\}\right) \\ &= \delta_{\nu,\nu'}\delta_{j,j'} + \delta_{\nu,\nu'}\sin\theta\cos\theta\left(\delta_{j,j'+1} + \delta_{j+1,j'}\right) \\ &= \delta_{\nu,\nu'}\left(\delta_{j,j'} + \frac{1}{2}\sin(2\theta)\left(\delta_{j,j'+1} + \delta_{j+1,j'}\right)\right).\end{aligned}$$

Proof 8.24 of Theorem 4.9

For the concrete form of x one calculates

$$\begin{aligned}
x_{\nu,q} &= \frac{1}{\sqrt{M}} \sum_j \mathrm{e}^{\mathrm{i}qx_j} x_{\nu,j} \\
&= \frac{1}{\sqrt{M}} \sum_j \mathrm{e}^{\mathrm{i}qx_j} \left(\frac{\cos\theta}{\sqrt{2}} \left((-1)^{\nu+1} a_{1,j+1} + a_{2,j} \right) + \frac{\sin\theta}{\sqrt{2}} \left((-1)^{\nu+1} a_{1,j} + a_{2,j+1} \right) \right) \\
&= (-1)^{\nu+1} \cdot \frac{\cos\theta}{\sqrt{2}} \left(\frac{1}{\sqrt{M}} \sum_j \mathrm{e}^{\mathrm{i}qx_{j+1}} a_{1,j+1} \right) \mathrm{e}^{-\mathrm{i}qa} + \frac{\cos\theta}{\sqrt{2}} \left(\frac{1}{\sqrt{M}} \sum_j \mathrm{e}^{\mathrm{i}qx_j} a_{2,j} \right) \\
&\quad + (-1)^{\nu+1} \cdot \frac{\sin\theta}{\sqrt{2}} \left(\frac{1}{\sqrt{M}} \sum_j \mathrm{e}^{\mathrm{i}qx_j} a_{1,j} \right) + \frac{\sin\theta}{\sqrt{2}} \left(\frac{1}{\sqrt{M}} \sum_j \mathrm{e}^{\mathrm{i}qx_{j+1}} a_{2,j+1} \right) \mathrm{e}^{-\mathrm{i}qa} \\
&= (-1)^{\nu+1} \frac{\cos\theta}{\sqrt{2}} a_{1,q} \mathrm{e}^{-\mathrm{i}qa} + \frac{\cos\theta}{\sqrt{2}} a_{2,q} + (-1)^{\nu+1} \frac{\sin\theta}{\sqrt{2}} a_{1,q} + \frac{\sin\theta}{\sqrt{2}} a_{2,q} \mathrm{e}^{-\mathrm{i}qa} \\
&= (-1)^{\nu+1} \frac{a_{1,q}}{\sqrt{2}} \left(\cos\theta \mathrm{e}^{-\mathrm{i}qa} + \sin\theta \right) + \frac{a_{2,q}}{\sqrt{2}} \left(\cos\theta + \sin\theta \mathrm{e}^{-\mathrm{i}qa} \right) \\
&=: (-1)^{\nu+1} c_1(q,\theta)\, a_{1,q} + c_2(q,\theta)\, a_{2,q}.
\end{aligned}$$

For the commutator one uses Theorem 4.8:

$$\begin{aligned}
\{x_{\nu,q}\,,\, x^\dagger_{\nu',p}\} &= \frac{1}{M} \sum_{j,j'} \mathrm{e}^{\mathrm{i}(qx_j - px_{j'})} \{x_{\nu,j}\,,\, x^\dagger_{\nu',j'}\} \\
&\overset{\text{Theorem 4.8}}{=} \frac{\delta_{\nu,\nu'}}{M} \sum_j \left(\mathrm{e}^{\mathrm{i}(p-q)x_j} + \sin\theta\cos\theta \left(\mathrm{e}^{\mathrm{i}(qx_j - px_{j+1})} + \mathrm{e}^{\mathrm{i}(qx_j - px_{j-1})} \right) \right) \\
&= \delta_{\nu,\nu'} \delta_{q,p} \left(1 + 2\cos\theta \sin\theta \cos(pa) \right) \\
&= \delta_{\nu,\nu'} \delta_{q,q'} \left(1 + \sin(2\theta)\cos(qa) \right) \\
&= \delta_{\nu,\nu'} \delta_{q,q'}\, N(q,\theta)^2.
\end{aligned}$$

Proof 8.25 of Remark 4.10.1

The first and third line follow immediately with Theorem 4.9. For the identities in the fourth line we compute with the inverse-rule for 2×2-matrices

$$\begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix} = \begin{pmatrix} c_1(k,\theta) & c_2(k,\theta) \\ -c_1(k,\theta) & c_2(k,\theta) \end{pmatrix} \begin{pmatrix} a_{1,k} \\ a_{2,k} \end{pmatrix} \iff \begin{pmatrix} a_{1,k} \\ a_{2,k} \end{pmatrix} = \frac{1}{2c_1(k,\theta)c_2(k,\theta)} \begin{pmatrix} c_2(k,\theta) & -c_2(k,\theta) \\ c_1(k,\theta) & c_1(k,\theta) \end{pmatrix} \begin{pmatrix} a_{1,k} \\ a_{2,k} \end{pmatrix}.$$

Accordingly we get

$$f(\nu,k,\theta) = \frac{\delta_{\nu,1}}{2c_1} + \frac{\delta_{\nu,2}}{2c_2}. \tag{8.6}$$

If we now expand this further with the third line, we get

$$
\begin{aligned}
f(1,k,\theta) &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\cos\theta\, \mathrm{e}^{-\mathrm{i}ka} + \sin\theta} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{1}{(\cos\theta\cos ka + \sin\theta) - \mathrm{i}\sin ka\cos\theta} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\cos\theta\cos ka + \sin\theta + \mathrm{i}\sin ka\cos\theta}{1+\sin(2\theta)\cos ka}
\end{aligned}
\qquad
\begin{aligned}
f(2,k,\theta) &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\cos\theta + \sin\theta\, \mathrm{e}^{-\mathrm{i}ka}} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{1}{(\cos\theta + \sin\theta\cos ka) - \mathrm{i}\sin ka\sin\theta} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\cos\theta + \sin\theta\cos ka + \mathrm{i}\sin ka\sin\theta}{1+\sin(2\theta)\cos ka}.
\end{aligned}
$$

For the second line it suffices (because of eq. (8.6)) to show, that

$$|c_1|^2 = |c_2|^2.$$

And indeed, it holds that

$$
\begin{aligned}
|c_1|^2 = c_1^* c_1 &= \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}ka}\cos\theta + \sin\theta\right)^* \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}ka}\cos\theta + \sin\theta\right) \\
&= \frac{1}{\sqrt{2}}\left(\mathrm{e}^{+\mathrm{i}ka}\cos\theta + \sin\theta\right) \frac{1}{\sqrt{2}}\left(\mathrm{e}^{-\mathrm{i}ka}\cos\theta + \sin\theta\right) \cdot \mathrm{e}^{-\mathrm{i}ka+\mathrm{i}ka} \\
&= \frac{1}{\sqrt{2}}\left(\cos\theta + \mathrm{e}^{-\mathrm{i}ka}\sin\theta\right) \frac{1}{\sqrt{2}}\left(\cos\theta + \mathrm{e}^{+\mathrm{i}ka}\sin\theta\right) \\
&= c_2 c_2^* = |c_2|^2.
\end{aligned}
$$

Further it follows

$$|c_1|^2 = \frac{1}{2}\left(\mathrm{e}^{+\mathrm{i}ka}\cos\theta + \sin\theta\right)\left(\mathrm{e}^{-\mathrm{i}ka}\cos\theta + \sin\theta\right) = \frac{1}{2}\left(1 + \sin\theta\cos\theta(\mathrm{e}^{\mathrm{i}ka} + \mathrm{e}^{-\mathrm{i}ka})\right) = \frac{1}{2}\left(1+\sin(2\theta)\cos(ka)\right).$$

and with this also the last identity in the second line.

Proof 8.26 of Theorem 4.11

The identities (i) and (ii) are immediate. The identity (iii) follows by explicit computation:

$$
\begin{aligned}
|\mathrm{D}\rangle = \prod_i d_{2,i}^\dagger |\mathrm{vac.}\rangle &= \frac{1}{\sqrt{M}^M} \sum_{q_1,\ldots,q_M} \left(\prod_j \mathrm{e}^{-\mathrm{i}q_j x_j}\right) \prod_j d_{2,q_j}^\dagger |\mathrm{vac.}\rangle \\
&= \frac{1}{\sqrt{M}^M} \sum_{\sigma\in S_M} \left(\prod_j \mathrm{e}^{-\mathrm{i}q_{\sigma(j)} x_j}\right) \prod_j d_{2,q_{\sigma(j)}}^\dagger |\mathrm{vac.}\rangle \\
&= \left(\frac{1}{\sqrt{M}^M} \sum_{\sigma\in S_M} \mathrm{sgn}(\sigma) \prod_j \mathrm{e}^{-\mathrm{i}q_{\sigma(j)} x_j}\right) \prod_q d_{2,q}^\dagger |\mathrm{vac.}\rangle \\
&= \prod_q d_{2,q}^\dagger |\mathrm{vac.}\rangle.
\end{aligned}
$$

Proof 8.27 of the footnote on page 38.

With the definition of the matrices on page 81, the operators L were introduced in the template [15] as

$$L_\alpha = \begin{cases} a_i^\dagger \, E^{a,1} \, x_i & \text{for } n = 1 \\ a_i^{\mathrm{T}} \, E^{a,2} \, x_i^{\mathrm{h.c.}} & \text{for } n = 2 \end{cases} .$$

Because of

- $E^{a,1} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } a = 1 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \text{for } a = 2 \end{cases}$,
- $E^{a,2} = \begin{cases} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{for } a = 1 \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } a = 2 \end{cases}$,

it follows

- $E^{a,1} \begin{pmatrix} a_{1,i} \\ a_{2,i} \end{pmatrix} = \begin{cases} \begin{pmatrix} a_{1,i} \\ 0 \end{pmatrix} & \text{for } a = 1 \\ \begin{pmatrix} 0 \\ a_{1,i} \end{pmatrix} & \text{for } a = 2 \end{cases}$,
- $E^{a,2} \begin{pmatrix} a_{1,i}^\dagger \\ a_{2,i}^\dagger \end{pmatrix} = \begin{cases} \begin{pmatrix} a_{2,i}^\dagger \\ 0 \end{pmatrix} & \text{for } a = 1 \\ \begin{pmatrix} 0 \\ a_{2,i}^\dagger \end{pmatrix} & \text{for } a = 2 \end{cases}$,

and with this it immediately follows that

$$\begin{aligned} L_{i,a,n} &= \sqrt{\kappa} \left(a_i^\dagger \, E^{a,1} x_i \delta_{n,1} + a_i^{\mathrm{T}} E^{a,2} x_i^{\mathrm{h.c.}} \delta_{n,2} \right) \\ &= \sqrt{\kappa} \left(\delta_{n,1} \left(\delta_{a,1} a_{1,i}^\dagger x_{1,i} + \delta_{a,2} a_{2,i}^\dagger x_{1,i} \right) + \delta_{n,2} \left(\delta_{a,1} a_{1,i} x_{2,i}^\dagger + \delta_{a,2} a_{2,i} x_{2,i}^\dagger \right) \right) \end{aligned}$$

coinciding with our definition.

8.1.3 Proofs to Chapter 5

Proof 8.28 of Lemma 5.4

We compute

$$\begin{aligned}
2\sum_{\alpha} L_\alpha \rho L_\alpha^\dagger = 2\kappa \sum_i \sum_{a,n} & \left(\delta_{n,1}\left(\delta_{a,1} a_{1,i}^\dagger x_{1,i} + \delta_{a,2} a_{2,i}^\dagger x_{1,i}\right) + \delta_{n,2}\left(\delta_{a,1} a_{1,i} x_{2,i}^\dagger + \delta_{a,2} a_{2,i} x_{2,i}^\dagger\right)\right) \rho \times \\
& \left(\delta_{n,1}\left(\delta_{a,1} x_{1,i}^\dagger a_{1,i} + \delta_{a,2} x_{1,i}^\dagger a_{2,i}\right) + \delta_{n,2}\left(\delta_{a,1} x_{2,i} a_{1,i}^\dagger + \delta_{a,2} x_{2,i} a_{2,i}^\dagger\right)\right) \\
= 2\kappa \sum_i \sum_{\nu\in\{1,2\}} & a_{\nu,i}^\dagger x_{1,i}\, \rho\, x_{1,i}^\dagger a_{\nu,i} \quad + \quad a_{\nu,i} x_{2,i}^\dagger\, \rho\, x_{2,i} a_{\nu,i}^\dagger \\
= 2\kappa \sum_{\nu\in\{1,2\}} \sum_i & a_{\nu,i}^\dagger x_{1,i}\, \rho\, x_{1,i}^\dagger a_{\nu,i} \quad + \quad a_{\nu,i} x_{2,i}^\dagger\, \rho\, x_{2,i} a_{\nu,i}^\dagger \\
= \frac{2\kappa}{N} \sum_{\nu\in\{1,2\}} \sum_{p_1,p_2,p_3,p_4} & \delta_{p_1+p_3-(p_2+p_4)} \left(a_{\nu,p_1}^\dagger x_{1,p_2}\, \rho\, x_{1,p_3}^\dagger a_{\nu,p_4} \quad + \quad a_{\nu,p_1} x_{2,p_2}^\dagger\, \rho\, x_{2,p_3} a_{\nu,p_4}^\dagger\right).
\end{aligned}$$

Using now the formula from Remark 4.10.1 gives immediately

- $a_{\nu,p_1}^\dagger x_{1,p_2}\, \rho\, x_{1,p_3}^\dagger a_{\nu,p_4} = f^*(\nu,p_1) f(\nu,p_4)\{(x_{1,p_1}^\dagger + (-1)^\nu x_{2,p_1}^\dagger) x_{1,p_2}\, \rho\, x_{1,p_3}^\dagger (x_{1,p_4} + (-1)^\nu x_{2,p_4})\}$,
- $a_{\nu,p_1} x_{2,p_2}^\dagger\, \rho\, x_{2,p_3} a_{\nu,p_4}^\dagger = f(\nu,p_1) f^*(\nu,p_4)\{(x_{1,p_1} + (-1)^\nu x_{2,p_1}) x_{2,p_2}^\dagger\, \rho\, x_{2,p_3} (x_{1,p_4}^\dagger + (-1)^\nu x_{2,p_4}^\dagger)\}$,

which, after dissolving the parenthesis, reproduces exactly the terms that were to be found.

Proof 8.29 of Lemma 5.5

We evaluate the terms $\mathrm{I}_j(\nu)$ for $(j,\nu) \in \{1,\dots,8\} \times \{1,2\}$. The procedure is divided into several steps:

1. In each quadruple sum $\sum_{p_1,p_2,p_3,p_4}$ we consider quadruples $(p_1,\dots,p_4)$ which contain $+p$ with different multiplicities[8].
2. In "wise foresight" we consider each of the terms $\mathrm{I}_{1,\dots,4}(\nu)$ and $\mathrm{I}_{5,\dots,8}(\nu)$ separately.
3. We perform simplifications obtained by the special shape of the summands in $\mathrm{I}_{1,\dots,4}(\nu)$ and $\mathrm{I}_{5,\dots,8}(\nu)$.
4. We evaluate the non-vanishing terms.
5. We merge the results.

[8] Those terms, which are constant, linear or cubic in the creation- and annihilation-operators (of the mode p) do not yield nontrivial contributions after taking the trace; The quartic terms are neglected in the framework of our approximation.

Index	p_1	p_2	p_3	p_4
2	p	p	-	-
3	p	-	p	-
4	p	-	-	p
5	-	p	p	-
6	-	p	-	p
7	-	-	p	p

Table 8.1: Quadruples $(p_1, \dots, p_4)$ that contain $+p$ with quadratic multiplicity.

$$\xrightarrow{\delta_{p_1+p_3-(p_2+p_4)}}$$

Index	p_1	p_2	p_3	p_4
2	p	p	p_4	-
3	p	$2p-p_4$	p	-
4	p	p_3	-	p
5	p_4	p	p	-
6	$2p-p_3$	p	-	p
7	p_2	-	p	p

Table 8.2: Quadruples $(p_1, \dots, p_4)$ that contain $+p$ with quadratic multiplicity and (inclusive) satisfy $p_1+p_3-(p_2+p_4)=0$.

For $\mathrm{I}_1(\nu), \dots, \mathrm{I}_4(\nu)$ further simplifications come up: Each contribution $\mathrm{I}_{1,\dots,4}(\nu)$ contains respective summands of the form „Factor“ $\cdot\, x^\dagger_{a,p_1} x_{1,p_2}\, \rho\, x^\dagger_{1,p_3} x_{b,p_4}$ with $(a,b) \in \{1,2\}^2$. This eliminates many contributions:

Index	p_1	p_2	p_3	p_4	$2\,\mathrm{Tr}_{\neq p}\, \delta_{p_1+p_3-(p_2+p_4)} f^*(\nu,p_1,\theta) f(\nu,p_4,\theta) x^\dagger_{a,p_1} x_{1,p_2}\, \rho\, x^\dagger_{1,p_3} x_{b,p_4}$
2	p	p	p_4	-	
3	p	$2p-p_4$	p	-	$=0$ since $\langle x_{b,p_4} x_{1,2p-p_4}\rangle = 0$
4	p	p_3	-	p	$=0$ since $\langle x^\dagger_{1,p_3} x_{1,p_3}\rangle = 0$
5	p_4	p	p	-	
6	$2p-p_3$	p	-	p	$=0$ since $\langle x^\dagger_{1,p_3} x^\dagger_{a,2p-p_3}\rangle = 0$
7	p_2	-	p	p	

Table 8.3: Quadruple-configurations which (anyway) do not contribute in $\mathrm{I}_{1,\dots,4}(\nu)$.

The remaining terms can be evaluated quickly:

Index	p_1	p_2	p_3	p_4	$2\,\mathrm{Tr}_{\neq p}\, \delta_{p_1+p_3-(p_2+p_4)} f^*(\nu,p_1,\theta) f(\nu,p_4,\theta) x^\dagger_{1,p_1} x_{1,p_2}\, \rho\, x^\dagger_{1,p_3} x_{1,p_4}$
2	p	p	p_4	-	$=0$ since $\langle x^\dagger_{1,p_4} x_{1,p_4}\rangle = 0$
5	p_4	p	p	-	$\neq 0$ since $\langle x_{1,p_4} x^\dagger_{1,p_4}\rangle \neq 0$
7	p_2	-	p	p	$=0$ since $\langle x^\dagger_{1,p_2} x_{1,p_2}\rangle = 0$

Table 8.4: Evaluation of $\mathrm{I}_1(\nu = 1,2)$.

Index	p_1	p_2	p_3	p_4	$2\,\mathrm{Tr}_{\neq p}\, \delta_{p_1+p_3-(p_2+p_4)} f^*(\nu,p_1,\theta) f(\nu,p_4,\theta) x^\dagger_{1,p_1} x_{1,p_2}\, \rho\, x^\dagger_{1,p_3} x_{2,p_4}$
2	p	p	p_4	-	$=0$ since $\langle x^\dagger_{1,p_4} x_{2,p_4}\rangle = 0$
5	p_4	p	p	-	$=0$ since $\langle x_{2,p_4} x^\dagger_{1,p_4}\rangle = 0$
7	p_2	-	p	p	$=0$ since $\langle x^\dagger_{1,p_2} x_{1,p_2}\rangle = 0$

Table 8.5: Evaluation of $\mathrm{I}_2(\nu = 1,2)$.

Index	p_1	p_2	p_3	p_4	$2\,\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f^*(\nu,p_1,\theta) f(\nu,p_4,\theta) x^\dagger_{2,p_1} x_{1,p_2}\,\rho\, x^\dagger_{1,p_3} x_{1,p_4}$
2	p	p	p_4	-	$= 0 \quad \text{since } \langle x^\dagger_{1,p_4} x_{1,p_4}\rangle = 0$
5	p_4	p	p	-	$= 0 \quad \text{since } \langle x_{1,p_4} x^\dagger_{2,p_4}\rangle = 0$
7	p_2	-	p	p	$= 0 \quad \text{since } \langle x^\dagger_{2,p_2} x_{1,p_2}\rangle = 0$

Table 8.6: Evaluation of $\mathrm{I}_3(\nu = 1, 2)$.

Index	p_1	p_2	p_3	p_4	$2\,\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f^*(\nu,p_1,\theta) f(\nu,p_4,\theta) x^\dagger_{2,p_1} x_{1,p_2}\,\rho\, x^\dagger_{1,p_3} x_{2,p_4}$
2	p	p	p_4	-	$= 0 \quad \text{since } \langle x^\dagger_{1,p_4} x_{2,p_4}\rangle = 0$
5	p_4	p	p	-	$= 0 \quad \text{since } \langle x_{2,p_4} x^\dagger_{2,p_4}\rangle = 0$
7	p_2	-	p	p	$= 0 \quad \text{since } \langle x^\dagger_{2,p_2} x_{1,p_2}\rangle = 0$

Table 8.7: Evaluation of $\mathrm{I}_4(\nu = 1, 2)$.

Since

$$\mathrm{I}_1(\nu = 1, 2) = \frac{\kappa}{M}\left(\sum_{k\neq p} 1\right)\cdot\, x_{1,p}\,\rho_p\, x^\dagger_{1,p} \approx \mathrm{K}(p,\theta)\, d_{1,p}\,\rho_p\, d^\dagger_{1,p},$$

the first part of the statement is proven.

For $\mathrm{I}_5(\nu)$, …, $\mathrm{I}_8(\nu)$, we follow the same procedure and observe the further simplifications: Each contribution $\mathrm{I}_{5,\ldots,8}(\nu)$ contains summands of the form „Factor“$\cdot\, x_{a,p_1} x^\dagger_{2,p_2}\,\rho\, x_{2,p_3} x^\dagger_{b,p_4}$ with $(a, b) \in \{1, 2\}^2$. This eliminates many other contributions:

Index	p_1	p_2	p_3	p_4	$2\,\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f(\nu,p_1,\theta) f^*(\nu,p_4,\theta) x_{a,p_1} x^\dagger_{2,p_2}\,\rho\, x_{2,p_3} x^\dagger_{b,p_4}$
2	p	p	p_4	-	
3	p	$2p - p_4$	p	-	$= 0 \quad \text{since } \langle x^\dagger_{b,p_4} x^\dagger_{2,2p-p_4}\rangle = 0$
4	p	p_3	-	p	$= 0 \quad \text{since } \langle x_{2,p_3} x^\dagger_{2,p_3}\rangle = 0$
5	p_4	p	p	-	
6	$2p - p_3$	p	-	p	$= 0 \quad \text{since } \langle x_{2,p_3} x_{a,2p-p_3}\rangle = 0$
7	p_2	-	p	p	

Table 8.8: Quadruple-configurations which (anyway) do not contribute in $\mathrm{I}_{5,\ldots,8}(\nu)$.

The remaining terms can again be evaluated quickly:

Index	p_1	p_2	p_3	p_4	$2\,\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f(\nu,p_1,\theta) f^*(\nu,p_4,\theta) x_{1,p_1} x^\dagger_{2,p_2}\,\rho\, x_{2,p_3} x^\dagger_{1,p_4}$
2	p	p	p_4	-	$= 0 \quad \text{since } \langle x_{2,p_4} x^\dagger_{1,p_4}\rangle = 0$
5	p_4	p	p	-	$= 0 \quad \text{since } \langle x^\dagger_{1,p_4} x_{1,p_4}\rangle = 0$
7	p_2	-	p	p	$= 0 \quad \text{since } \langle x_{1,p_2} x^\dagger_{2,p_2}\rangle = 0$

Table 8.9: Evaluation of $\mathrm{I}_5(\nu = 1, 2)$.

Index	p_1	p_2	p_3	p_4	$2\,\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f(\nu,p_1,\theta) f^*(\nu,p_4,\theta) x_{1,p_1} x^\dagger_{2,p_2}\,\rho\, x_{2,p_3} x^\dagger_{2,p_4}$
2	p	p	p_4	-	$= 0$ since $\langle x_{2,p_4} x^\dagger_{2,p_4}\rangle = 0$
5	p_4	p	p	-	$= 0$ since $\langle x^\dagger_{2,p_4} x_{1,p_4}\rangle = 0$
7	p_2	-	p	p	$= 0$ since $\langle x_{1,p_2} x^\dagger_{2,p_2}\rangle = 0$

Table 8.10: Evaluation of $\mathrm{I}_6(\nu = 1, 2)$.

Index	p_1	p_2	p_3	p_4	$2\,\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f(\nu,p_1,\theta) f^*(\nu,p_4,\theta) x_{2,p_1} x^\dagger_{2,p_2}\,\rho\, x_{2,p_3} x^\dagger_{1,p_4}$
2	p	p	p_4	-	$= 0$ since $\langle x_{2,p_4} x^\dagger_{1,p_4}\rangle = 0$
5	p_4	p	p	-	$= 0$ since $\langle x^\dagger_{1,p_4} x_{2,p_4}\rangle = 0$
7	p_2	-	p	p	$= 0$ since $\langle x_{2,p_2} x^\dagger_{2,p_2}\rangle = 0$

Table 8.11: Evaluation of $\mathrm{I}_7(\nu = 1, 2)$.

Index	p_1	p_2	p_3	p_4	$2\,\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f(\nu,p_1,\theta) f^*(\nu,p_4,\theta) x_{2,p_1} x^\dagger_{2,p_2}\,\rho\, x_{2,p_3} x^\dagger_{b,p_4}$
2	p	p	p_4	-	$= 0$ since $\langle x_{2,p_4} x^\dagger_{2,p_4}\rangle = 0$
5	p_4	p	p	-	$\neq 0$ since $\langle x^\dagger_{2,p_4} x_{2,p_4}\rangle \neq 0$
7	p_2	-	p	p	$= 0$ since $\langle x_{2,p_2} x^\dagger_{2,p_2}\rangle = 0$

Table 8.12: Evaluation of $\mathrm{I}_8(\nu = 1, 2)$.

Since

$$\mathrm{I}_8(\nu = 1, 2) = \frac{\kappa}{M}\left(\sum_{k\neq p} 1\right) \cdot x^\dagger_{2,p}\,\rho_p\, x_{2,p} \approx \mathrm{K}(p,\theta)\, d^\dagger_{2,p}\,\rho_p\, d_{2,p},$$

the statement is proven.

Proof 8.30 of Lemma 5.7

We evaluate the terms in complete analogy to Lemma 5.5 and follow the five steps in the above proof.

For $\mathrm{II}_1(\nu)$, …, $\mathrm{II}_4(\nu)$ one again has further simplifications: Each contribution $\mathrm{II}_{1,\dots,4}(\nu)$ contains respective summands of the form „Factor“ $\cdot\, x^\dagger_{1,p_1} x_{a,p_2} x^\dagger_{b,p_3} x_{1,p_4}\,\rho$ with $(a,b) \in \{1,2\}^2$. This eliminates many contributions:

Index	p_1	p_2	p_3	p_4	$\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f(\nu,p_2,\theta) f^*(\nu,p_3,\theta) x^\dagger_{1,p_1} x_{a,p_2} x^\dagger_{b,p_3} x_{1,p_4}\,\rho$
2	p	p	p_4	-	$= 0$ since $\langle x^\dagger_{b,p_4} x_{1,p_4}\rangle = 0$
3	p	$2p-p_4$	p	-	$= 0$ since $\langle x_{a,2p-p_4} x_{1,p_4}\rangle = 0$
4	p	p_3	-	p	
5	p_4	p	p	-	$= 0$ since $\langle x^\dagger_{1,p_4} x^\dagger_{1,p_4}\rangle = 0$
6	$2p-p_3$	p	-	p	$= 0$ since $\langle x^\dagger_{1,2p-p_3} x^\dagger_{b,p_3}\rangle = 0$
7	p_2	-	p	p	$= 0$ since $\langle x^\dagger_{1,p_2} x_{a,p_2}\rangle = 0$

Table 8.13: Quadruple-configurations which (anyway) do not contribute in $\mathrm{II}_{1,\dots,4}(\nu)$.

The remaining terms can again be evaluated quickly:

Index	p_1	p_2	p_3	p_4	$\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f(\nu,p_2,\theta) f^*(\nu,p_3,\theta) x^\dagger_{1,p_1} x_{1,p_2} x^\dagger_{1,p_3} x_{1,p_4}\,\rho$
4	p	p_3	-	p	$\neq 0 \quad$ since $\langle x_{1,p_3} x^\dagger_{1,p_3}\rangle \neq 0$

Table 8.14: Evaluation of $\mathrm{II}_1(\nu)$.

Index	p_1	p_2	p_3	p_4	$\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f(\nu,p_2,\theta) f^*(\nu,p_3,\theta) x^\dagger_{1,p_1} x_{2,p_2} x^\dagger_{1,p_3} x_{1,p_4}\,\rho$
4	p	p_3	-	p	$= 0 \quad$ since $\langle x_{2,p_3} x^\dagger_{1,p_3}\rangle = 0$

Table 8.15: Evaluations of $\mathrm{II}_2(\nu)$.

Index	p_1	p_2	p_3	p_4	$\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f(\nu,p_2,\theta) f^*(\nu,p_3,\theta) x^\dagger_{1,p_1} x_{1,p_2} x^\dagger_{2,p_3} x_{1,p_4}\,\rho$
4	p	p_3	-	p	$= 0 \quad$ since $\langle x_{1,p_3} x^\dagger_{2,p_3}\rangle = 0$

Table 8.16: Evaluation of $\mathrm{II}_3(\nu)$.

Index	p_1	p_2	p_3	p_4	$\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f(\nu,p_2,\theta) f^*(\nu,p_3,\theta) x^\dagger_{1,p_1} x_{2,p_2} x^\dagger_{2,p_3} x_{1,p_4}\,\rho$
4	p	p_3	-	p	$= 0 \quad$ since $\langle x_{2,p_3} x^\dagger_{2,p_3}\rangle = 0$

Table 8.17: Evaluation of $\mathrm{II}_4(\nu)$.

Since

$$\mathrm{II}_1(\nu=1,2) = \frac{\kappa}{2M}\left(\sum_{k\neq p} 1\right)\cdot x^\dagger_{1,p} x_{1,p}\,\rho_p = \frac{\mathrm{K}(p,\theta)}{2}\, d^\dagger_{1,p} d_{1,p}\,\rho_p,$$

the first part of the statement is proven.

For $\mathrm{II}_5(\nu),\ldots,\mathrm{II}_8(\nu)$ we follow the same procedure and observe further simplifications: Each contribution $\mathrm{II}_{5,\ldots,8}(\nu)$ contains summands of the form „Factor“ $\cdot\, x_{2,p_1} x^\dagger_{a,p_2} x_{b,p_3} x^\dagger_{2,p_4}\,\rho$ with $(a,b)\in\{1,2\}^2$. This eliminates many other contributions:

Index	p_1	p_2	p_3	p_4	$\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f^*(\nu,p_2,\theta) f(\nu,p_3,\theta) x_{2,p_1} x^\dagger_{a,p_2} x_{b,p_3} x^\dagger_{2,p_4}\rho$
2	p	p	p_4	-	$= 0 \quad$ since $\langle x_{b,p_4} x^\dagger_{2,p_4}\rangle = 0$
3	p	$2p-p_4$	p	-	$= 0 \quad$ since $\langle x^\dagger_{a,2p-p_4} x^\dagger_{2,p_4}\rangle = 0$
4	p	p_3	-	p	
5	p_4	p	p	-	$= 0 \quad$ since $\langle x_{2,p_4} x^\dagger_{2,p_4}\rangle = 0$
6	$2p-p_3$	p	-	p	$= 0 \quad$ since $\langle x_{2,2p-p_3} x_{b,p_3}\rangle = 0$
7	p_2	-	p	p	$= 0 \quad$ since $\langle x_{2,p_2} x^\dagger_{a,p_2}\rangle = 0$

Table 8.18: Quadruple-configurations which (anyway) do not contribute in $\mathrm{II}_{5,\ldots,8}(\nu)$.

The remaining terms can again be evaluated quickly:

Index	p_1	p_2	p_3	p_4	$\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f^*(\nu,p_2,\theta) f(\nu,p_3,\theta) x_{2,p_1} x^\dagger_{1,p_2} x_{1,p_3} x^\dagger_{2,p_4} \rho$
4	p	p_3	-	p	$= 0 \quad$ since $\langle x^\dagger_{1,p_3} x_{1,p_3} \rangle = 0$

Table 8.19: Evaluation of $\mathrm{II}_5(\nu)$.

Index	p_1	p_2	p_3	p_4	$\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f^*(\nu,p_2,\theta) f(\nu,p_3,\theta) x_{2,p_1} x^\dagger_{2,p_2} x_{1,p_3} x^\dagger_{2,p_4} \rho$
4	p	p_3	-	p	$= 0 \quad$ since $\langle x^\dagger_{2,p_3} x_{1,p_3} \rangle = 0$

Table 8.20: Evaluation of $\mathrm{II}_6(\nu)$.

Index	p_1	p_2	p_3	p_4	$\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f^*(\nu,p_2,\theta) f(\nu,p_3,\theta) x_{2,p_1} x^\dagger_{1,p_2} x_{2,p_3} x^\dagger_{2,p_4} \rho$
4	p	p_3	-	p	$= 0 \quad$ since $\langle x^\dagger_{1,p_3} x_{2,p_3} \rangle = 0$

Table 8.21: Evaluation of $\mathrm{II}_7(\nu)$.

Index	p_1	p_2	p_3	p_4	$\mathrm{Tr}_{\neq p}\,\delta_{p_1+p_3-(p_2+p_4)} f^*(\nu,p_2,\theta) f(\nu,p_3,\theta) x_{2,p_1} x^\dagger_{2,p_2} x_{2,p_3} x^\dagger_{2,p_4} \rho$
4	p	p_3	-	p	$\neq 0 \quad$ since $\langle x^\dagger_{2,p_3} x_{2,p_3} \rangle = 0$

Table 8.22: Evaluation of $\mathrm{II}_8(\nu)$.

Since

$$\mathrm{II}_8(\nu = 1,2) = \frac{\kappa}{2M} \left(\sum_{k \neq p} 1 \right) \cdot x_{2,p} x^\dagger_{2,p}\, \rho_p = \frac{\mathrm{K}(p,\theta)}{2}\, d_{2,p} d^\dagger_{2,p}\, \rho_p$$

the statement is proven.

Proof 8.31 of Approximation 5.9

As mentioned earlier, the basic idea is to approximate quartic terms in the action by weighted bilinears. To illustrate this concept we consider the following identity (in schematic notation):

According to standard-literature as [1], [18], it holds that

$$\int \prod_j \mathrm{d}\psi_j \exp - \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j = \det A$$

for an arbitrary, complex-valued matrix A. Applying Wick's-theorem then gives[9]

$$\begin{aligned} \langle \psi_a \overline{\psi}_b \rangle &= A^{\mathrm{inv}}_{ab} \\ \langle \psi_a \psi_b \overline{\psi}_c \overline{\psi}_d \rangle &= -A^{\mathrm{inv}}_{ac} A^{\mathrm{inv}}_{bd} + A^{\mathrm{inv}}_{ad} A^{\mathrm{inv}}_{bc}, \end{aligned}$$

where

[9] With $A^{\mathrm{inv}}_{ab} \equiv \left(A^{\mathrm{inv}}\right)_{ab}$ etc.

$$\langle \psi_a \overline{\psi}_b \rangle \equiv \frac{\int \prod_j \mathrm{d}\psi_j \, \psi_a \overline{\psi}_b \exp - \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j}{\int \prod_j \mathrm{d}\psi_j \exp - \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j}$$

and therefore one has in particular

$$\begin{aligned} \langle \psi_a \psi_b \overline{\psi}_c \overline{\psi}_d \rangle &= -\langle \psi_a \overline{\psi}_c \rangle A^{\mathrm{inv}}_{bd} + \langle \psi_a \overline{\psi}_d \rangle A^{\mathrm{inv}}_{bc} \\ &= \langle -\psi_a \overline{\psi}_c A^{\mathrm{inv}}_{bd} + \psi_a \overline{\psi}_d A^{\mathrm{inv}}_{bc} \rangle. \end{aligned}$$

This allows the series of equations

$$\begin{aligned} &\int \prod_j \mathrm{d}\psi_j \exp\Big[- \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j \Big] \, \exp\Big[z_1 z_2 \overline{z}_3 \overline{z}_4 \Big] \\ &= \int \prod_j \mathrm{d}\psi_j \exp\Big[- \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j \Big] \, \left(1 + z_1 z_2 \overline{z}_3 \overline{z}_4 \right) \\ &= \int \prod_j \mathrm{d}\psi_j \exp\Big[- \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j \Big] \, \left(1 + z_1 \overline{z}_3 \, A^{\mathrm{inv}}_{z_2 \overline{z}_4} - z_1 \overline{z}_4 \, A^{\mathrm{inv}}_{z_2 \overline{z}_3} \right) \\ &= \int \prod_j \mathrm{d}\psi_j \exp\Big[- \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j \Big] \, \exp\Big[z_1 \overline{z}_3 \, A^{\mathrm{inv}}_{z_2 \overline{z}_4} - z_1 \overline{z}_4 \, A^{\mathrm{inv}}_{z_2 \overline{z}_3} \Big]. \end{aligned}$$

For two quartic terms one can now do the following approximation:

$$\begin{aligned} &\int \prod_j \mathrm{d}\psi_j \exp\Big[- \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j \Big] \, \exp\Big[y_1 y_2 \overline{y}_3 \overline{y}_4 + z_1 z_2 \overline{z}_3 \overline{z}_4 \Big] \\ &\overset{\mathrm{BCH}}{=} \int \prod_j \mathrm{d}\psi_j \exp\Big[- \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j \Big] \, \exp\Big[y_1 y_2 \overline{y}_3 \overline{y}_4 \Big] \, \exp\Big[z_1 z_2 \overline{z}_3 \overline{z}_4 \Big] \\ &= \int \prod_j \mathrm{d}\psi_j \exp\Big[- \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j \Big] \, \left(1 + y_1 y_2 \overline{y}_3 \overline{y}_4 \right) \, \left(1 + z_1 z_2 \overline{z}_3 \overline{z}_4 \right) \\ &\approx \int \prod_j \mathrm{d}\psi_j \exp\Big[- \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j \Big] \, \left(1 + y_1 \overline{y}_3 \, A^{\mathrm{inv}}_{y_2 \overline{y}_4} - y_1 \overline{y}_4 \, A^{\mathrm{inv}}_{y_2 \overline{y}_3} \right) \left(1 + z_1 \overline{z}_3 \, A^{\mathrm{inv}}_{z_2 \overline{z}_4} - z_1 \overline{z}_4 \, A^{\mathrm{inv}}_{z_2 \overline{z}_3} \right) \\ &\overset{\mathrm{BCH}}{=} \int \prod_j \mathrm{d}\psi_j \exp\Big[- \sum_{i,j} \overline{\psi}_i A_{i,j} \psi_j \Big] \, \exp\Big[y_1 \overline{y}_3 \, A^{\mathrm{inv}}_{y_2 \overline{y}_4} - y_1 \overline{y}_4 \, A^{\mathrm{inv}}_{y_2 \overline{y}_3} + z_1 \overline{z}_3 \, A^{\mathrm{inv}}_{z_2 \overline{z}_4} - z_1 \overline{z}_4 \, A^{\mathrm{inv}}_{z_2 \overline{z}_3} \Big], \end{aligned}$$

in which some correlators (those between y and $\overline{z}$, and those between z and $\overline{y}$) are neglected.

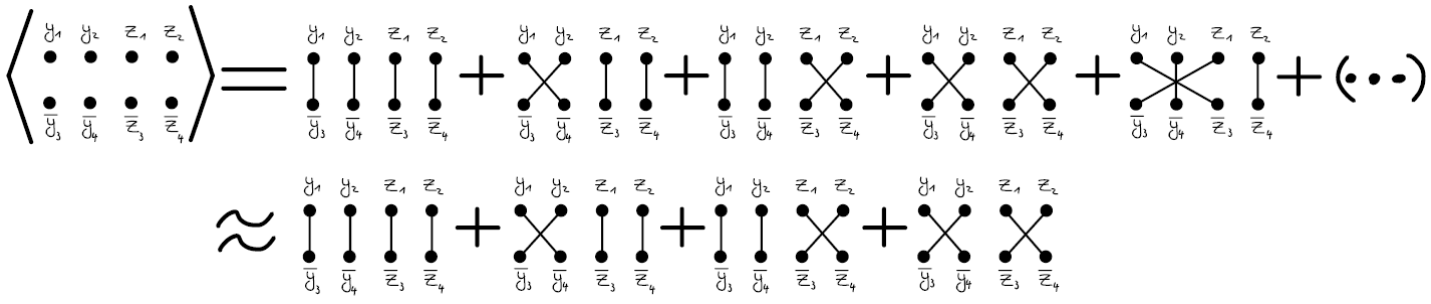


Figure 8.1: The step involving an approximation in above equation (schematically): Some correlators are being neglected.

Thus, following the above principle, one can perform these approximations for general functional integrals such as those studied in this thesis. Schematically:

$$\begin{aligned}&\exp\{\sum \text{“Quadratic Terms”} + \sum \text{“Quartic Terms”}\}\\ \approx\ &\exp\{\sum \text{“Quadratic Terms”} + \sum \text{“More Quadratic Terms”}\}.\end{aligned}$$

In the approximation of [17], one evaluates the resulting correlators not with respect to the matrix A, but with respect to the dark state, which is a solution of the stationary Lindblad-equation. In this sense, the approximation is called “self-consistent”.

The concrete case

In our situation, the interaction term of the action reads

$$\begin{aligned}S_{\text{int}}[\overline{\psi},\psi] &= \kappa \oint_{t,\alpha} 2L_{+,\alpha}L^{\dagger}_{-,\alpha} - \left((L^{\dagger}L)_{+,\alpha} + (L^{\dagger}L)_{-,\alpha}\right)\\ &= \kappa \sum_{t_n} \Delta t \sum_{\alpha} \mathrm{I} + \mathrm{II} + \mathrm{III}\,,\end{aligned}$$

where (neglecting boundary terms)

$$\begin{aligned}&\bullet\ \mathrm{I} \quad = 2\frac{\langle\psi_{+,n+1}|L_{\alpha}|\psi_{+,n}\rangle\langle-\psi_{-,n}|L^{\dagger}_{\alpha}|-\psi_{-,n+1}\rangle}{\langle\psi_{+,n+1}|\psi_{+,n}\rangle\langle-\psi_{-,n}|-\psi_{-,n+1}\rangle} = 2L_{\alpha}(\overline{\psi}_{+,n+1},\psi_{+,n})L^{\dagger}_{\alpha}(-\overline{\psi}_{-,n},-\psi_{-,n+1}),\\ &\bullet\ \mathrm{II} \quad = -\frac{\langle\psi_{+,n+1}|L^{\dagger}_{\alpha}L_{\alpha}|\psi_{+,n}\rangle}{\langle\psi_{+,n+1}|\psi_{+,n}\rangle} = -\left(L^{\dagger}_{\alpha}L_{\alpha}\right)(\overline{\psi}_{+,n+1},\psi_{+,n}),\\ &\bullet\ \mathrm{III} = -\frac{\langle-\psi_{-,n}|L^{\dagger}_{\alpha}L_{\alpha}|\psi_{-,n+1}\rangle}{\langle-\psi_{-,n}|\psi_{-,n+1}\rangle} = -\left(L^{\dagger}_{\alpha}L_{\alpha}\right)(-\overline{\psi}_{-,n},-\psi_{-,n+1}).\end{aligned}$$

Now for each $\alpha = (j,\nu,\mu) \in \{1,\dots,M\}\times\{1,2\}\times\{1,2\}$ we obtain an expression that is quartic, e.g. for II one has a contribution from $\nu = \mu = 1$ that is

$$\begin{aligned}&-\sum_{j=1}^{M}\frac{\langle\psi_{+,n+1}|x^{\dagger}_{1,j}a_{1,j}a^{\dagger}_{1,j}x_{1,j}|\psi_{+,n}\rangle}{\langle\psi_{+,n+1}|\psi_{+,n}\rangle}\\ &= -\frac{1}{M}\sum_{q_1,\dots,q_4}\delta(q_1+q_3-(q_2+q_4))f(1,q_2,\theta)f^{*}(1,q_3,\theta)\frac{\langle\psi_{+,n+1}|x^{\dagger}_{1,q_1}(x_{1,q_2}-x_{2,q_2})(x^{\dagger}_{1,q_3}-x^{\dagger}_{2,q_3})x_{1,q_4}|\psi_{+,n}\rangle}{\langle\psi_{+,n+1}|\psi_{+,n}\rangle}.\end{aligned}$$

Avoiding the normal ordering by introducing additional timestamps as in [17] gives four terms. These can be approximated in the sense above (by means of a change of variables $\psi \to \chi$) to give

$$-\frac{\delta_{q_2,q_3}}{2|f(q_2,\theta)|^2}\chi_{1,+}(t,q_4)\overline{\chi}_{1,+}(q_1,t) + 0 + 0 + 0,$$

compensating the respective f-valued prefactors. The control results are

$$
\begin{aligned}
\bullet\,\mathrm{I} &= 2\sum_q \overline{\chi}_{1,-}(t,q)\chi_{1,+}(t,q) - \overline{\chi}_{2,+}(t,q)\chi_{2,-}(t,q),\\
\bullet\,\mathrm{II} &= \sum_q -\overline{\chi}_{1,+}(t,q)\chi_{1,+}(t,q) + \overline{\chi}_{2,+}(t,q)\chi_{2,+}(t,q),\\
\bullet\,\mathrm{III} &= \sum_q -\overline{\chi}_{1,-}(t,q)\chi_{1,-}(t,q) + \overline{\chi}_{2,-}(t,q)\chi_{2,-}(t,q).
\end{aligned}
$$

Rescaling to $\chi \to \xi/\sqrt{2}|f|$ gives the desired result.

Proof 8.32 of Theorem 5.10

We follow the strategy in [18]. For the first band and a mode q, we obtain the inverse propagator in discrete notation

$$
\int \mathrm{d}t \begin{pmatrix} \overline{\xi}_{+,1} \\ \overline{\xi}_{-,1} \end{pmatrix}^{\mathrm{T}} \cdot \begin{pmatrix} \overleftarrow{\partial}_t - \mathrm{K}(q,\theta) & 0 \\ 2\,\mathrm{K}(q,\theta) & \partial_t - \mathrm{K}(q,\theta) \end{pmatrix} \cdot \begin{pmatrix} \xi_{+,1} \\ \xi_{-,1} \end{pmatrix} \equiv \sum_{n,n'} \sum_{\nu,\nu' \in \{-,+\}} \overline{\xi}_{\nu,1}(q,t_n)\, G^{-1}_{0,1}(q,\nu,\nu',t_n,t_{n'})\, \xi_{\nu',1}(q,t_{n'}),
$$

with a $2(N+1)\times 2(N+1)$-matrix $G^{-1}_{0,1}(q,\nu,\nu',t,t') \equiv G^{-1}_{0,1}(q,\nu,\nu',t_n,t_{n'})$, such that one arrives for $N=3$ at the matrix

$$
G^{-1}_{0,1}(q,\nu,\nu',t_n,t_m) = \left(\begin{array}{cccc|cccc}
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
A & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A & -1 & 0 & 0 & 0 & 0 \\
\hline
B & 0 & 0 & 0 & -1 & A & 0 & 0 \\
0 & B & 0 & 0 & 0 & -1 & A & 0 \\
0 & 0 & B & 0 & 0 & 0 & -1 & A \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1
\end{array}\right)_{\nu,\nu',n,n'},
$$

where $A = 1 + \Delta t\ \mathrm{K}(q,\theta)$ and $B = \Delta t\,\mathrm{K}(q,\theta)$. This matrix can be inverted analytically to

$$
G_{0,1}(q,\nu,\nu',t_n,t_{n'}) \equiv \begin{pmatrix} G^{\mathbb{T}}_{0,1}(q,t_n,t_{n'}) & G^{<}_{0,1}(q,t_n,t_{n'}) \\ G^{>}_{0,1}(q,t_n,t_{n'}) & G^{\widetilde{\mathbb{T}}}_{0,1}(q,t_n,t_{n'}) \end{pmatrix}_{\nu,\nu'},
$$

where

$$G^{\mathbb{T}}_{0,1}(q,t_n,t_{n'}) = \frac{1}{\operatorname{Det} G_{0,1}(q)} \begin{pmatrix} -1 & A(A^4+B+A^2B) & A^2(A^2+B) & A^3 \\ -A & -1-B & A^3(A^2+B) & A^4 \\ -A^2 & -A(1+B) & -1-(1+A^2)B & A^5 \\ -A^3 & -A^2(1+B) & -A(1+B+A^2B) & -1-(1+A^2+A^4)B \end{pmatrix}_{n,n'},$$

$$G^{<}_{0,1}(q,t_n,t_{n'}) = \frac{1}{\operatorname{Det} G_{0,1}(q)} \begin{pmatrix} 1 & A & A^2 & A^3 \\ A & A^2 & A^3 & A^4 \\ A^2 & A^3 & A^4 & A^5 \\ A^3 & A^4 & A^5 & A^6 \end{pmatrix}_{n,n'},$$

$$G^{>}_{0,1}(q,t_n,t_{n'}) = \frac{1}{\operatorname{Det} G_{0,1}(q)} \begin{pmatrix} -1\cdot(A^6+(1+A+A^4)B) & -A\cdot 1\cdot(A^4+B+A^2B) & -A^2(A^2+B)\cdot 1 & -A^3 \\ -A\cdot 1\cdot(A^4+B+A^2B) & -(1+B)(A^4+(1+A^2)B) & -A(1+B)(A^2+B) & A^2\cdot 1\cdot(1+B) \\ -A^2(A^2+B)\cdot 1 & -A(1+B)(A^2+B) & -(1+(1+A^2)B)(A^2+B) & -A(1+B+A^2B)\cdot 1 \\ -A^3 & -A^2\cdot 1\cdot(1+B) & -A(1+B+A^2B)\cdot 1 & -(1+(1+A^2+A^4)B)\cdot 1 \end{pmatrix}_{n,n'},$$

$$G^{\widetilde{\mathbb{T}}}_{0,1}(q,t_n,t_{n'}) = \frac{1}{\operatorname{Det} G_{0,1}(q)} \begin{pmatrix} -1 & -A & -A^2 & -A^3 \\ A(A^4+B+A^2B) & -1-B & -A(1+B) & -A^2(1+B) \\ A^2(A^2+B) & A^3(A^2+B) & -1-(1+A^2)B & -A(1+B+A^2B) \\ A^3 & A^4 & A^5 & -1-(1+A^2+A^4)B \end{pmatrix}_{n,n'},$$

and

$$\operatorname{Det} G_{0,1}(q) = 1 + A^6 + (1+A^2+A^4)B.$$

Analogously to [18] one can compute the generalization of the $N=3$-example to arbitrary $N \in \mathbb{N}_+$, by using the finite geometric series[10] and using further

$$\frac{B}{1-A^2} = \frac{\Delta t\, \mathrm{K}(q,\theta)}{1-(1-\Delta t\, \mathrm{K}(q,\theta))^2} = \frac{1}{1+\mathcal{O}(\Delta t)}$$

to be

$$\operatorname{Det} G_{0,1}(q) = 1 + A^{2N} + (1+A^2+\ldots+A^{2(N-1)})B = 1 + A^{2N} + \frac{1-A^{2N}}{1-A^2}B = 2 + \mathcal{O}(\Delta t)$$

and e.g.

$$\begin{aligned} G^{\mathbb{T}}_{0,1}(q,t_n,t_{n'}) &= \frac{1}{2}\begin{cases} -A^{n-n'}\left(1+B\sum_{k=0}^{n'-1}A^{2k}\right) & \text{for } n \geq n' \\ A^{2N-(n'-n)} + A^{2n+(n'-n)}\left(\sum_{k=0}^{N-(1+n')}A^{2k}\right)B & \text{for } n < n' \end{cases} \\ &= \ldots \\ &= \frac{\theta(t_n-t_{n'})}{\exp\left(|t_n-t_{n'}|\,\mathrm{K}(q,\theta)\right)} - \frac{1}{2\exp\left(|t_n+t_{n'}|\,\mathrm{K}(q,\theta)\right)} \end{aligned}$$

and analogous identities and calculations for the remaining correlators.

Proof 8.33 of Corollary 5.12 For the first band one computes

[10] By a clever multiplication of a one, one obtains a telescoping series

$$\sum_{k=0}^{n} q^k = \left(1+q+q^2+\ldots+q^n\right)\cdot\frac{1-q}{1-q} = \frac{1-q^{n+1}}{1-q}.$$

$$\begin{pmatrix} g(\theta) & 0 \\ -(g+g^*)(\theta) & g^*(\theta) \end{pmatrix}^{\text{inv}} = \frac{1}{|g(\theta)|^2} \cdot \begin{pmatrix} g^*(\theta) & 0 \\ (g+g^*)(\theta) & g(\theta) \end{pmatrix},$$

and for the second band one has

$$\begin{pmatrix} -g^*(\theta) & (g+g^*)(\theta) \\ 0 & -g(\theta) \end{pmatrix}^{\text{inv}} = \frac{1}{|g(\theta)|^2} \cdot \begin{pmatrix} -g(\theta) & -(g+g^*)(\theta) \\ 0 & -g^*(\theta) \end{pmatrix}.$$

With this the corollary is proven.

Proof 8.34 of Lemma 5.16

The function

$$\mathbb{C} \ni \omega \mapsto \frac{1}{g(\omega,q,\theta)} \cdot \frac{1}{g^*(\omega+\Delta\omega,q+\Delta q,\theta)} = \frac{1}{\mathrm{i}\omega + \mathrm{K}(q,\theta)} \cdot \frac{1}{-\mathrm{i}(\omega+\Delta\omega) + \mathrm{K}(q+\Delta q,\theta)} \equiv f(\omega) \in \mathbb{C}$$

possesses simple poles at $\omega \in \{+\mathrm{i}\,\mathrm{K}(q,\theta), -\mathrm{i}\,\mathrm{K}(q+\Delta q,\theta) - \Delta\omega\}$. With the residual theorem (it holds that $K(q,\theta) \geq 0$) we obtain, with γ denoting the path in the figure,

$$\int \frac{\mathrm{d}\omega}{2\pi} f(\omega) = \frac{1}{2\pi} \oint_\gamma \mathrm{d}z\, f(z) = \mathrm{i}\,\mathrm{Res}(f, \mathrm{i}\,\mathrm{K}(q,\theta)) = \frac{1}{-\mathrm{i}\Delta\omega + \mathrm{K}(q,\theta)) + \mathrm{K}(q+\Delta q,\theta)}.$$

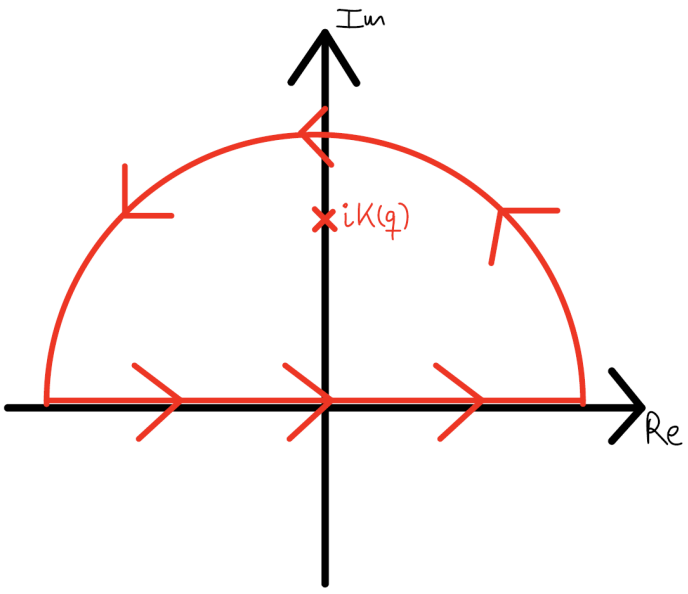


Figure 8.2: Visualization of the curve γ over which is integrated.

Proof 8.35 of Lemma 5.18.

The first two identities follow immediately with eq. (5.5). For the (anti-)symmetry of the argument we first note with eq. (5.1)

$$\mathrm{K}(-q) = \kappa\,(1 + \sin(2\theta)\cos(-qa)) = \kappa\,(1 + \sin(2\theta)\cos(qa)) = \mathrm{K}(q).$$

And since

$$\sum_q f(q) = \sum_q f(-q),$$

one concludes the equation

$$\begin{aligned}
\mathrm{Re}\, D(-\Delta\,\mathrm{Q}) &= \frac{2\kappa}{M}\frac{1}{2^2}\sum_q \frac{\mathrm{K}(q) + \mathrm{K}(q-\Delta q)}{(\Delta\omega)^2 + (\mathrm{K}(q) + \mathrm{K}(q-\Delta q))^2} \\
&= \frac{2\kappa}{M}\frac{1}{2^2}\sum_q \frac{\mathrm{K}(-q) + \mathrm{K}(-(q+\Delta q))}{(\Delta\omega)^2 + (\mathrm{K}(-q) + \mathrm{K}(-(q+\Delta q)))^2} \\
&= \frac{2\kappa}{M}\frac{1}{2^2}\sum_q \frac{\mathrm{K}(q) + \mathrm{K}(q+\Delta q)}{(\Delta\omega)^2 + (\mathrm{K}(q) + \mathrm{K}(q+\Delta q))^2} \\
&= \mathrm{Re}\, D(\Delta\,\mathrm{Q}),
\end{aligned}$$

$$\begin{aligned}
\mathrm{Im}\, D(-\Delta\,\mathrm{Q}) &= \frac{2\kappa}{M}\frac{1}{2^2}\sum_q \frac{-\Delta\omega}{(\Delta\omega)^2 + (\mathrm{K}(q) + \mathrm{K}(q-\Delta q))^2} \\
&= -\frac{2\kappa}{M}\frac{1}{2^2}\sum_q \frac{\Delta\omega}{(\Delta\omega)^2 + (\mathrm{K}(-q) + \mathrm{K}(-(q+\Delta q)))^2} \\
&= -\frac{2\kappa}{M}\frac{1}{2^2}\sum_q \frac{\mathrm{K}(q) + \mathrm{K}(q+\Delta q)}{(\Delta\omega)^2 + (\mathrm{K}(q) + \mathrm{K}(q+\Delta q))^2} \\
&= -\,\mathrm{Im}\, D(\Delta\,\mathrm{Q}).
\end{aligned}$$

For the infrared-divergence we consider the $\Delta\,\mathrm{Q} = 0$-sector and decompose the area of integration with $\epsilon > 0$ according to

$$[0, \frac{2\pi}{a}) = [0, \frac{\pi}{a} - \epsilon) \cup (\frac{\pi}{a} - \epsilon\,,\, \frac{\pi}{a} + \epsilon) \cup (\frac{\pi}{a} + \epsilon, \frac{2\pi}{a}).$$

Then, it follows

$$\mathrm{K}(q = \frac{\pi}{a} \pm \delta) \;\approx\; \frac{1}{2}\delta^2 \quad \text{für} \quad \delta \in (-\epsilon, \epsilon)$$

and with this one calculates

$$
\begin{aligned}
D(\Delta\,\mathrm{Q}=0) \propto \frac{1}{M}\sum_q \frac{1}{\mathrm{K}(q)} &\approx \int \frac{\mathrm{d}q}{2\pi}\frac{1}{\mathrm{K}(q)} \\
&= \int_0^{\frac{\pi}{a}-\epsilon} \frac{\mathrm{d}q}{2\pi}\frac{1}{\mathrm{K}(q)} + \int_{\frac{\pi}{a}+\epsilon}^{\frac{2\pi}{a}} \frac{\mathrm{d}q}{2\pi}\frac{1}{\mathrm{K}(q)} + \int_{\frac{\pi}{a}-\epsilon}^{\frac{\pi}{a}+\epsilon} \frac{\mathrm{d}q}{2\pi}\frac{1}{\mathrm{K}(q)} \\
&\approx \qquad\qquad \text{const.} \qquad\qquad + \int_{-\epsilon}^{+\epsilon} \mathrm{d}\delta\,\frac{1}{\delta^2}.
\end{aligned}
$$

Alternatively one immediately sees that in the sum $\sum_q \frac{1}{\mathrm{K}(q)}$, there is a "1/0" (division by zero) for $q=\frac{\pi}{a}$.

Proof 8.36 of Theorem 5.20.

The basic idea is to show that a loop-term L in eq. (5.3) cannot diverge if $\mathscr{F}$ in eq. (5.4) is at least quadratic in $1/f$.

We first consider those quadruples $(n_1,n_2,n_3,n_4)\equiv n$ that contain at least two zeros. Because of the symmetry of $\mathscr{F}(q,\Delta q,n_1,n_2,n_3,n_4)\equiv\mathscr{F}(q,\Delta q,n)$ and since we are in the $\Delta\,\mathrm{Q}=0$, $\theta=\frac{\pi}{4}$-sector, it is sufficient to consider the three cases $n=(0,0,1,1)$, $n=(0,0,1,2)$ and $n=(0,0,2,2)$.

Case 1:

For $n=(0,0,1,1)$ we have with eq. (5.4) and Remark (4.10.1)

$$
\mathscr{F}(q,0=\Delta q,0,0,1,1)=\frac{1\cdot 1\cdot f^*(1,q+0)f(1,q+0)}{|f(q)|^2|f(q+0)|^2}=\frac{1}{|f(q)|^2}=2\,(1+\cos(qa))
$$

such that the divergence in the loop-contribution is cured because then one has

$$
\frac{\mathscr{F}(q,0=\Delta q,0,0,1,1)}{\mathrm{i}\cdot 0+(\mathrm{K}(q)+\mathrm{K}(q+0))}\propto 1\neq\infty \quad \text{for any value of } q,
$$

and therefore one no longer divides by zero for $q=\frac{\pi}{a}$.

Case 2:

For $n=(0,0,1,2)$, because

$$
\frac{f^*(1,q,\theta)f(2,q,\theta)}{|f(q,\theta)|^2}\overset{\theta=\pi/4}{=}1
$$

we have again

$$
\mathscr{F}(q,0=\Delta q,0,0,1,2)\propto\mathrm{K}(q),
$$

such that no diverging terms can arise.

Case 3:

For $n=(0,0,2,2)$ we obtain with Remark (4.10.1) a calculation analogous to case 1.

Because of the symmetry of $\mathscr{F}$ the further configurations of n with exactly two, three or four zeros

$$n \equiv (0,0,2,1),(0,1,1,0),(0,1,2,0),(0,2,1,0),(0,2,2,0),(1,1,0,0),(1,2,0,0) \quad \text{etc.}$$

do not generate divergent corrections.

We now consider the quadruples for the function $\mathscr{F}$ that have exactly one zero. Therefore we consider the sets $\mathcal{M}_{1,2,3,4}$, where

$\mathcal{M}_1 = \{(0,1,1,1), (1,0,1,1), (1,1,0,1), (1,1,1,0)\}$,
$\mathcal{M}_2 = \{(0,1,1,2), (0,1,2,1), (0,2,1,1), (1,0,1,2), (1,0,2,1), (1,1,0,2), (1,1,2,0), (1,2,0,1), (1,2,1,0), (2,0,1,1), (2,1,0,1), (2,1,1,0)\}$,
$\mathcal{M}_2 = \{(0,2,2,1), (0,2,1,2), (0,1,2,2), (2,0,2,1), (2,0,1,2), (2,2,0,1), (2,2,1,0), (2,1,0,2), (2,1,2,0), (1,0,2,2), (1,2,0,2), (1,2,2,0)\}$,
$\mathcal{M}_4 = \{(0,2,2,2),(2,0,2,2),(2,2,0,2),(2,2,2,0)\}$.

For these quadruples $\mathscr{F}$ cannot cure a divergence in a loop-contribution. It holds, e. g.,

$$\mathscr{F}(q,0=\Delta q,0,1,1,1) \propto \frac{1}{f^*(1,q)} \propto \frac{1+\cos(qa)}{1+\cos(qa)-\mathrm{i}\sin(qa)} \overset{q/a=\pi}{=} 1$$

but the terms that correspond to these quadruples for the prefactor $\mathscr{F}$ miraculously cancel out in the $\mathrm{Q}_1 = \mathrm{Q}_2 = \mathrm{Q}_3 = \mathrm{Q}_4$-sector (in the sector, in which the divergences are generated) which was shown by the numerical[11] calculation.

With Lemma 5.15 and Lemma 5.19 the statement is now proven.

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8.2 Tables for some Vertex-Functions

Theorem 8.1 With the abbreviations

$$C(i,q_j,q_k) \equiv \frac{1}{2} f^*(i,q_j) f(i,q_k), \qquad \Xi \equiv \Xi(\mathrm{Q}_1,\dots,\mathrm{Q}_4) \equiv \frac{2\kappa}{M} \frac{\delta(\mathrm{Q}_1+\mathrm{Q}_2-(\mathrm{Q}_3+\mathrm{Q}_4))}{2^2 \prod_{i=1}^4 |f(q_i)|},$$

where $f(i,q) \equiv f(i,q,\theta)$ is as in Remark 4.10.1, and

$$\delta(\mathrm{Q}_1+\mathrm{Q}_2-(\mathrm{Q}_3+\mathrm{Q}_4)) \equiv \delta(\omega_1+\omega_2-(\omega_3+\omega_4))\,\delta_{q_1+q_2-(q_3+q_4)}$$

we obtain the following table for particular vertex-functions.

Remark 8.1.1 The number of quartic vertex-functions being exactly 100 is a mathematical accident. It is easy to derive this number: Consider a monomial of degree 4 from the series expansion of the effective action (e.g. a term from the equation in Theorem 5.14):

$$\overline{\xi}_{I_1}(\mathrm{Q}_1)\,\overline{\xi}_{I_2}(\mathrm{Q}_2)\,\xi_{I_3}(\mathrm{Q}_3)\,\xi_{I_4}(\mathrm{Q}_4), \tag{8.7}$$

with $I_j \in \{+,-\} \times \{1,2\}$ for $j \in \{1,\dots,4\}$. As mentioned in Remark 3.12.1, one usually avoids the double-counting for discrete indices while one allows the double-counting for continuous indices. Due to the symmetry one must now only look for the number of linearly independent monomials of the type

$$\xi_{I_3}(\mathrm{Q}_3)\,\xi_{I_4}(\mathrm{Q}_4) \tag{8.8}$$

for a fixed momentum-configuration (satisfying $\mathrm{Q}_3 \neq \mathrm{Q}_4$) which is visualized in Figure 7.2 pictographically. This number is 10. Therefore one has in total $10 \cdot 10 = 100$ different monomials of the shape in eq. (8.7).

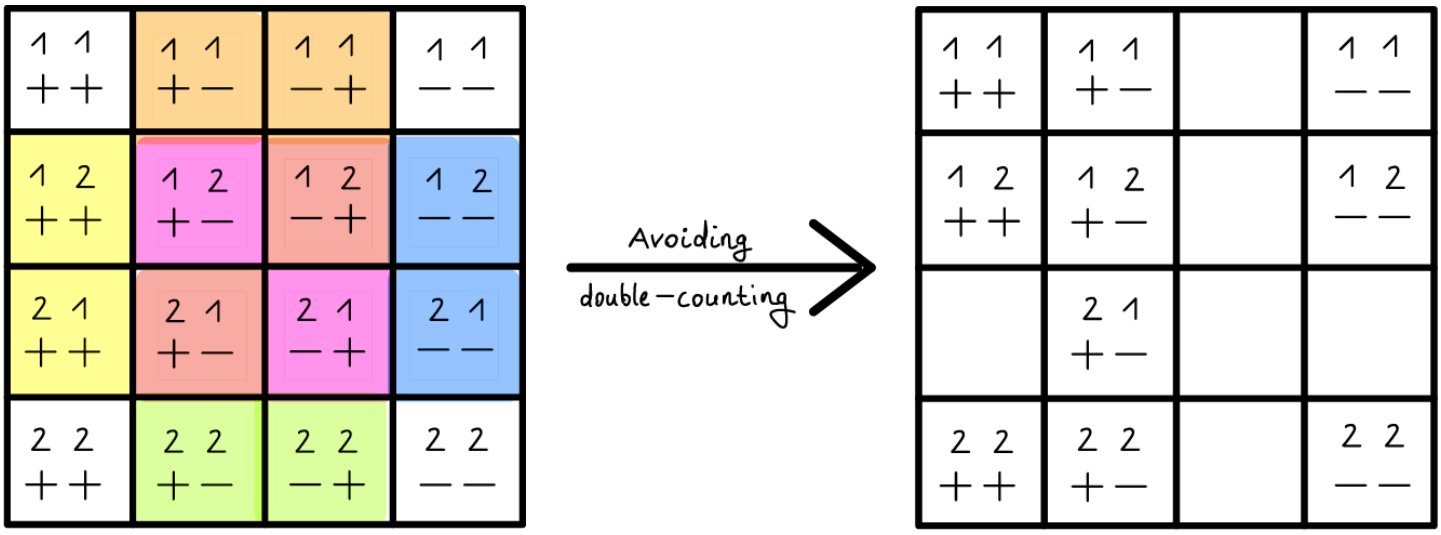


Figure 8.3: Visualization of the counting scheme: monomials in eq. (8.7) correspond to a quadruple being built up from two pairs. All possible pairs (allowing double counting) are depicted left, and two pairs of the same color are linearly dependent on each other, because of the commutation relations of the Graßmann-numbers. The right part of the figure therefore counts all possible pairs (avoiding double counting).

Remark 8.1.2 One can easily calculate, that the sanity-check from Corollary 3.6 is fulfilled for the Tree-Level-approximation. One easily calculates

For the 1111**-Quadruples:**

$$\left(\gamma_{\text{Tree},0} + \gamma_{\text{Tree},33} + \gamma_{\text{Tree},66}\right)_{|_{(+,-)\mapsto(+,+)}} = 0.$$

For the 1112**-Quadruples:**

$$\left(\gamma_{\text{Tree},1} + \gamma_{\text{Tree},34} + \gamma_{\text{Tree},67}\right)_{|_{(+,-)\mapsto(+,+)}} = 0.$$

For the 1211**-Quadruples:**

$$\left(\gamma_{\text{Tree},10} + \gamma_{\text{Tree},43} + \gamma_{\text{Tree},96}\right)_{|_{(+,-)\mapsto(+,+)}} = 0.$$

For the 1212**-Quadruples:**

$$\left(\gamma_{\text{Tree},11} + \gamma_{\text{Tree},44} + \gamma_{\text{Tree},97}\right)_{|_{(+,-)\mapsto(+,+)}} = 0.$$

For the 1222**-Quadruples:**

$$\left(\gamma_{\text{Tree},12} + \gamma_{\text{Tree},45} + \gamma_{\text{Tree},98}\right)_{|_{(+,-)\mapsto(+,+)}} = 0.$$

For the 2212**-Quadruples:**

$$\left(\gamma_{\text{Tree},21} + \gamma_{\text{Tree},54} + \gamma_{\text{Tree},87}\right)_{|_{(+,-)\mapsto(+,+)}} = 0.$$

For the 2222**-Quadruples:**

$$\left(\gamma_{\text{Tree},22} + \gamma_{\text{Tree},55} + \gamma_{\text{Tree},88}\right)_{|_{(+,-)\mapsto(+,+)}} = 0.$$

Index	Vertex-Function	In Γ_{Tree}	In Γ_{∞}
0	$\gamma^{1111}_{++++}(Q_1,\dots,Q_4)$	$\Xi\, C(1,Q_2,Q_3)+\Xi\, C(2,Q_2,Q_3)$	0
1	$\gamma^{1112}_{++++}(Q_1,\dots,Q_4)$	$\Xi\, C(1,Q_2,Q_4)-\Xi\, C(2,Q_2,Q_4)$	0
2	$\gamma^{1122}_{++++}(Q_1,\dots,Q_4)$	0	0
3	$\gamma^{1111}_{++--}(Q_1,\dots,Q_4)$	0	0
4	$\gamma^{1112}_{++--}(Q_1,\dots,Q_4)$	0	0
5	$\gamma^{1122}_{++--}(Q_1,\dots,Q_4)$	0	0
6	$\gamma^{1111}_{+++-}(Q_1,\dots,Q_4)$	0	0
7	$\gamma^{1112}_{+++-}(Q_1,\dots,Q_4)$	0	0
8	$\gamma^{1122}_{+++-}(Q_1,\dots,Q_4)$	0	0
9	$\gamma^{1112}_{++-+}(Q_1,\dots,Q_4)$	0	0
10	$\gamma^{1211}_{++++}(Q_1,\dots,Q_4)$	$-\Xi\, C(1,Q_2,Q_3)+\Xi\, C(2,Q_2,Q_3)$	0
11	$\gamma^{1212}_{++++}(Q_1,\dots,Q_4)$	$-\sum_{i=1}^{2}\Xi\, C(i,Q_2,Q_4)+\Xi\, C(i,Q_1,Q_3)$	0
12	$\gamma^{1222}_{++++}(Q_1,\dots,Q_4)$	$-\Xi\, C(1,Q_1,Q_4)+\Xi\, C(2,Q_1,Q_4)$	0
13	$\gamma^{1211}_{++--}(Q_1,\dots,Q_4)$	0	0
14	$\gamma^{1212}_{++--}(Q_1,\dots,Q_4)$	0	0
15	$\gamma^{1222}_{++--}(Q_1,\dots,Q_4)$	0	0
16	$\gamma^{1211}_{+++-}(Q_1,\dots,Q_4)$	0	0
17	$\gamma^{1212}_{+++-}(Q_1,\dots,Q_4)$	0	$4\,\Xi\, D(Q_2,Q_4)$
18	$\gamma^{1222}_{+++-}(Q_1,\dots,Q_4)$	0	0
19	$\gamma^{1212}_{++-+}(Q_1,\dots,Q_4)$	0	0
20	$\gamma^{2211}_{++++}(Q_1,\dots,Q_4)$	0	0
21	$\gamma^{2212}_{++++}(Q_1,\dots,Q_4)$	$\Xi\, C(1,Q_1,Q_3)-\Xi\, C(2,Q_1,Q_3)$	0
22	$\gamma^{2222}_{++++}(Q_1,\dots,Q_4)$	$\Xi\, C(1,Q_1,Q_4)+\Xi\, C(2,Q_1,Q_4)$	0
23	$\gamma^{2211}_{++--}(Q_1,\dots,Q_4)$	0	0
24	$\gamma^{2212}_{++--}(Q_1,\dots,Q_4)$	0	0
25	$\gamma^{2222}_{++--}(Q_1,\dots,Q_4)$	0	$-8\,\mathrm{Re}\;\Xi\, D(Q_2,Q_4)$
26	$\gamma^{2211}_{+++-}(Q_1,\dots,Q_4)$	0	0
27	$\gamma^{2212}_{+++-}(Q_1,\dots,Q_4)$	0	0
28	$\gamma^{2222}_{+++-}(Q_1,\dots,Q_4)$	0	$4\,\Xi\, D(Q_2,Q_4)$
29	$\gamma^{2212}_{++-+}(Q_1,\dots,Q_4)$	0	0
30	$\gamma^{1111}_{--++}(Q_1,\dots,Q_4)$	0	$-8\,\mathrm{Re}\;\Xi\, D(Q_2,Q_4)$
31	$\gamma^{1112}_{--++}(Q_1,\dots,Q_4)$	0	0
32	$\gamma^{1122}_{--++}(Q_1,\dots,Q_4)$	0	0
33	$\gamma^{1111}_{----}(Q_1,\dots,Q_4)$	$\Xi\, C(1,Q_2,Q_3)+\Xi\, C(2,Q_2,Q_3)$	0

Index	Vertex-Function	In Γ_{Tree}	In $\Gamma_{\text{1-loop, Div.}}$
34	$\gamma^{1112}_{----}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	$\Xi\, C(1,\mathrm{Q}_2,\mathrm{Q}_4) - \Xi\, C(2,\mathrm{Q}_2,\mathrm{Q}_4)$	0
35	$\gamma^{1122}_{----}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
36	$\gamma^{1111}_{--+-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	$4\,\Xi\, D^*(\mathrm{Q}_2,\mathrm{Q}_4)$
37	$\gamma^{1112}_{--+-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
38	$\gamma^{1122}_{--+-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
39	$\gamma^{1112}_{---+}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
40	$\gamma^{1211}_{--++}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
41	$\gamma^{1212}_{--++}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
42	$\gamma^{1222}_{--++}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
43	$\gamma^{1211}_{----}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	$-\Xi\, C(1,\mathrm{Q}_2,\mathrm{Q}_3) + \Xi\, C(2,\mathrm{Q}_2,\mathrm{Q}_3)$	0
44	$\gamma^{1212}_{----}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	$-\sum_{i=1}^{2} \Xi\, C(i,\mathrm{Q}_2,\mathrm{Q}_4) + \Xi\, C(i,\mathrm{Q}_1,\mathrm{Q}_3)$	0
45	$\gamma^{1222}_{----}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	$-\Xi\, C(1,\mathrm{Q}_1,\mathrm{Q}_4) + \Xi\, C(2,\mathrm{Q}_1,\mathrm{Q}_4)$	0
46	$\gamma^{1211}_{--+-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
47	$\gamma^{1212}_{--+-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	$4\,\Xi\, D^*(\mathrm{Q}_2,\mathrm{Q}_4)$
48	$\gamma^{1222}_{--+-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
49	$\gamma^{1212}_{---+}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
50	$\gamma^{2211}_{--++}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
51	$\gamma^{2212}_{--++}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
52	$\gamma^{2222}_{--++}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
53	$\gamma^{2211}_{----}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
54	$\gamma^{2212}_{----}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	$\Xi\, C(1,\mathrm{Q}_1,\mathrm{Q}_3) - \Xi\, C(2,\mathrm{Q}_1,\mathrm{Q}_3)$	0
55	$\gamma^{2222}_{----}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	$\Xi\, C(1,\mathrm{Q}_1,\mathrm{Q}_4) + \Xi\, C(2,\mathrm{Q}_1,\mathrm{Q}_4)$	0
56	$\gamma^{2211}_{--+-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
57	$\gamma^{2212}_{--+-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
58	$\gamma^{2222}_{--+-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
59	$\gamma^{2212}_{---+}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
60	$\gamma^{1111}_{+-++}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	$4\,\Xi\, D(\mathrm{Q}_2,\mathrm{Q}_4)$
61	$\gamma^{1112}_{+-++}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
62	$\gamma^{1122}_{+-++}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
63	$\gamma^{1111}_{+---}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
64	$\gamma^{1112}_{+---}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
65	$\gamma^{1122}_{+---}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	0	0
66	$\gamma^{1111}_{+-+-}(\mathrm{Q}_1,\dots,\mathrm{Q}_4)$	$-2\Xi\, C(1,\mathrm{Q}_1,\mathrm{Q}_4) - 2\Xi\, C(2,\mathrm{Q}_1,\mathrm{Q}_4)$	0

Index	Vertex-Function	In Γ_{Tree}	In $\Gamma_{\text{1-loop, Div.}}$
67	$\gamma^{1112}_{+-+-}(Q_1,\dots,Q_4)$	$2\,\Xi\,C(1,Q_1,Q_4) - 2\,\Xi\,C(2,Q_1,Q_4)$	0
68	$\gamma^{1122}_{+-+-}(Q_1,\dots,Q_4)$	0	0
69	$\gamma^{1112}_{+--+}(Q_1,\dots,Q_4)$	0	0
70	$\gamma^{1211}_{+-++}(Q_1,\dots,Q_4)$	0	0
71	$\gamma^{1212}_{+-++}(Q_1,\dots,Q_4)$	0	0
72	$\gamma^{1222}_{+-++}(Q_1,\dots,Q_4)$	0	0
73	$\gamma^{1211}_{+---}(Q_1,\dots,Q_4)$	0	0
74	$\gamma^{1212}_{+---}(Q_1,\dots,Q_4)$	0	0
75	$\gamma^{1222}_{+---}(Q_1,\dots,Q_4)$	0	0
76	$\gamma^{2211}_{+-+-}(Q_1,\dots,Q_4)$	0	0
77	$\gamma^{2212}_{+-+-}(Q_1,\dots,Q_4)$	0	0
78	$\gamma^{2222}_{+-+-}(Q_1,\dots,Q_4)$	0	0
79	$\gamma^{2212}_{+--+}(Q_1,\dots,Q_4)$	0	0
80	$\gamma^{2211}_{+-++}(Q_1,\dots,Q_4)$	0	0
81	$\gamma^{2212}_{+-++}(Q_1,\dots,Q_4)$	0	0
82	$\gamma^{2222}_{+-++}(Q_1,\dots,Q_4)$	0	0
83	$\gamma^{2211}_{+---}(Q_1,\dots,Q_4)$	0	0
84	$\gamma^{2212}_{+---}(Q_1,\dots,Q_4)$	0	0
85	$\gamma^{2222}_{+---}(Q_1,\dots,Q_4)$	0	$4\,\Xi\,D^*(Q_2,Q_4)$
86	$\gamma^{2211}_{+-+-}(Q_1,\dots,Q_4)$	0	0
87	$\gamma^{2212}_{+-+-}(Q_1,\dots,Q_4)$	$2\,\Xi\,C(1,Q_2,Q_3) - 2\,\Xi\,C(2,Q_2,Q_3)$	0
88	$\gamma^{2222}_{+-+-}(Q_1,\dots,Q_4)$	$-2\,\Xi\,C(1,Q_2,Q_3) - 2\,\Xi\,C(2,Q_2,Q_3)$	0
89	$\gamma^{2212}_{+--+}(Q_1,\dots,Q_4)$	0	0
90	$\gamma^{1211}_{-+++}(Q_1,\dots,Q_4)$	0	0
91	$\gamma^{1212}_{-+++}(Q_1,\dots,Q_4)$	0	$4\,\Xi\,D^*(Q_2,Q_4)$
92	$\gamma^{1222}_{-+++}(Q_1,\dots,Q_4)$	0	0
93	$\gamma^{1211}_{-+--}(Q_1,\dots,Q_4)$	0	0
94	$\gamma^{1212}_{-+--}(Q_1,\dots,Q_4)$	0	$4\,\Xi\,D(Q_2,Q_4)$
95	$\gamma^{1222}_{-+--}(Q_1,\dots,Q_4)$	0	0
96	$\gamma^{1211}_{-++-}(Q_1,\dots,Q_4)$	$-2\,\Xi\,C(1,Q_2,Q_4) + 2\,\Xi\,C(2,Q_2,Q_4)$	0
97	$\gamma^{1212}_{-++-}(Q_1,\dots,Q_4)$	$2\sum_{i=1}^{2}\Xi\,C(i,Q_2,Q_4) + \Xi\,C(i,Q_1,Q_3)$	$-16\,\mathrm{Re}\;\Xi\,D(Q_2,Q_4)$
98	$\gamma^{1222}_{-++-}(Q_1,\dots,Q_4)$	$-2\,\Xi\,C(1,Q_1,Q_3) + 2\,\Xi\,C(2,Q_1,Q_3)$	0
99	$\gamma^{1212}_{-+-+}(Q_1,\dots,Q_4)$	0	0

Bibliography

[1] A. Altland, B. D. Simons, *Condensed Matter Field Theory*: 2nd ed., Cambridge University Press, 2010.

[2] H.-P. Breuer, F. Petruccione, *The Theory of open quantum systems*, Oxford University Press, 2002.

[3] D. Manzano: "A short Introduction to the Lindblad Master Equation", arXiv:1906.04478, June 2020.

[4] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press, 2014.

[5] S. Diehl: Class Lecture, Topic: "Quantum Field Theory I, II"
University of Cologne, Cologne, 2020-2021.

[6] M. Scherer: Class Lecture, Topic: "Interacting Fermi Systems"
University of Cologne, Cologne, 2019-2020.

[7] R. Thomale, C. Platt and W. Hanke: "Functional Renormalization Group for multi-orbital Fermi Surface Instabilities", arXiv:1310.6191, October 2008.

[8] J. Braun, H. Gies, D. D. Scherer: "Asymptotic safety: a simple example", arXiv:1011.1456v1, November 2010.

[9] F. Tonielli, J. C. Budich, A. Altland and S. Diehl: "Topological Keldysh field theory and a Coleman-Hill theorem for driven open quantum systems", (in preparation).

[10] W. Metzner, M. Salmhofer, C. Honerkamp, V. Meden, K. Schönhammer: "Functional renormalization group approach to correlated fermion systems", *RMP*, vol. 84, p. 300-319, January - March, 2012.

[11] M. E. Peskin, D. V. Schroeder, *An Introduction to Quantum Field Theory*, CRC Press, 1995.

[12] F. Schütz, L. Bartosch, P. Kopietz, *Introduction to the Functional Renormalization Group*, Springer Spektrum, 2010.

[13] M. Zirnbauer: Class Lecture, Topic: "Advanced Quantum Mechanics"
University of Cologne, Cologne, 2010-2011.

[14] A. Cheung: Class Lecture, Topic: "Notes on Phase Transitions and Collective Phenomena"
University of Cambridge, Cavendish Laboratory, Cambridge, 2021.

[15] A. Altland, M. Fleischhauer and S. Diehl: "Symmetry Classes of Open Fermionic Quantum Matter", arXiv:2007.10448, July 2020.

[16] C.-E. Bardyn, M.A. Baranov, C.V. Kraus, E. Rico, A. Imamoglu, P. Zoller and S. Diehl: "Topology by Dissipation", arXiv:1302.5135, February 2013.

[17] F. Tonielli, J. C. Budich, A. Altland and S. Diehl: "Topological Field Theory far from Equilibrium", arXiv:1911.07834, November 2019.

[18] A. Kamenev, *Field Theory of Non-Equilibrium Systems*, Cambridge University Press, 2011.

[19] M.R. Zirnbauer: Class Lecture, Topic: "Quantum Field Theory I, II"
University of Cologne, Cologne, 2019-2020.

[20] T. R. Morris: "The exact renormalization group and approximate solutions", arXiv:hep-ph/9308265, October 1994.

[21] J. Berges, N. Tetradis, C. Wetterich: "Non-Perturbative Renormalization Flow in Quantum Field Theory Statistical Physics", arXiv:hep-ph/0005122, May 2000.

[22] M. Scherer: Class Lecture, Topic: "Introduction to Renormalization with Applications in Condensed-Matter and High-Energy Physics", University of Cologne, Cologne, 2017-2018.

[23] L. Ryder, *Quantum Field Theory*, 2nd ed., Cambride University Press, 1996.

[24] S. Kalhöfer, C. Lange, Y. Schaden, "Report on the Experiment β-Scintillation", University of Cologne, Cologne, Germany, 2019.

[25] S. Kalhöfer, J. Reuter, Y. Schaden, "Report on the Experiment Transport Properties of Copper", University of Cologne, Cologne, Germany, 2020.

[26] J. K. Asbóth, L. Oroszlány, A. P. Pályi: "A Short Course on Topological Insulators: Band Structure and Edge States in One and Two Dimensions", arXiv:1509.02295, September 2015.

[27] L. Li, C. Yang and S. Chen: "Winding numbers of phase transition points for one-dimensional topological systems", arXiv:1508.01680, August 2015.

[28] L. M. Sieberer, M. Buchhold, S. Diehl: "Keldysh Field Theory for Driven Open Quantum Systems", arXiv:1512.00637v2, August 2016.

Declaration

I hereby confirm that I have written this thesis independently and that I have not used any aids other than those indicated. The passages in the thesis that are taken from other works (including internet sources) in terms of wording or meaning have been marked with an indication of the source.

Date: _____ Signature: _____